

Math 365 – Monday 2/4/19
Section 2.3: Functions

Attach at the end of Homework 2:

At the end of your write-up, include the following, labeling this as “**Writing exercise**”.

- (a) Mark up your finished homework assignment, showing where you followed or failed to follow the mechanical and stylistic issues outlined in the handout *Communicating Mathematics through Homework and Exams*. This means **treat your write-up as a second-to-last draft**, and go point-by-point through the handout and address instances where you followed or did not follow each direction in your writing. Use a different-colored pen if you have one, and hand in this marked up draft. You do not need to rewrite the result.

How did you improve this week over homework 1? How might you improve in the future?

- (b) List three or more ways that you succeeded or failed at following the advice in *Some Guidelines for Good Mathematical Writing*. How did you improve this week over homework 1? How might you improve in the future?

To receive credit for this assignment, you must complete this exercise.

Exercise 8. (See last page for definition quick-reference)

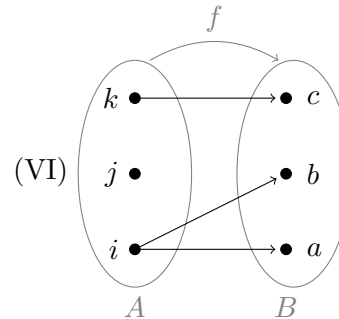
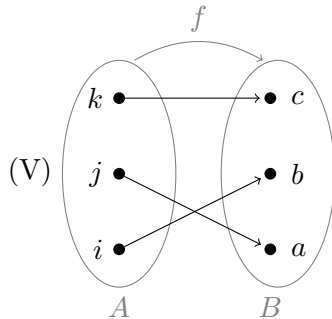
- (a) Decide for each of the following expressions: Is it a function? If so,
 (i) what is its domain, codomain, and image? (ii) is it injective? (why or why not)
 (iii) is it surjective? (why or why not) (iv) is it invertible? (why or why not)

(I) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $x \mapsto x^3$

(II) $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $x \mapsto \sqrt{x}$

(III) $f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}$ defined by $(a, b) \mapsto a/b$

(IV) $f : \mathbb{R} \times \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $(r, z) \mapsto [r] * z$



- (b) For those examples which were not functions in the previous problems, can you restrict the domain and/or codomain to make them functions? i.e. pick a reasonable subset of the given domain/codomain such that the expression is a function on that domain.

- (c) If possible, choose a domain and codomain for the following expressions to make them into functions satisfying

- (i) f is an injective but not surjective function;
 (ii) f is a surjective but not injective function;
 (iii) f is an invertible function;
 (iv) f is not a function;

where f is given by:

- (I) $f(x) = |x|$,
 (II) $f(x) = 1/x$,
 (III) $f(x) = 6$,
 (IV) $f(x, y) = \sqrt{xy}$.

- (d) Let f be a function from the set A to the set B . Let S and T be subsets of A . Show that
- $f(S \cup T) = f(S) \cup f(T)$; and
 - $f(S \cap T) \subseteq f(S) \cap f(T)$.

Can you think of an example where the inclusion in part (b) could be proper (not equal)?

Exercise 9. Let

$$f : A \rightarrow B \quad \text{and} \quad g : B \rightarrow C$$

be functions. Draw some pictures and make some conjectures about the following questions.

- Is $g \circ f$ always a function?
- What are the conditions on f , g , A , B , and/or C for $g \circ f$ to be surjective?
- What are the conditions on f , g , A , B , and/or C for $g \circ f$ to be injective?
- What are the conditions on f , g , A , B , and/or C for $g \circ f$ to be bijective?
- What are the conditions on f , g , A , B , and/or C for $f \circ g$ to be a function?
- If f and $g \circ f$ are injective, is it necessarily true that g is injective?
- If f and $g \circ f$ are surjective, is it necessarily true that g is surjective?

Exercise 10. Show that if both

$$f : A \rightarrow B \quad \text{and} \quad g : B \rightarrow C$$

are surjective functions, then $g \circ f$ is also surjective.

Exercise 11.

- Determine whether f is a function from \mathbb{Z} to \mathbb{R} if
 - $f(n) = \pm n$;
 - $f(n) = \sqrt{n^2 + 1}$;
 - $f(n) = 1/(n^2 - 4)$.
- Let $f(x) = 2x$ where the domain is the set of real numbers. Compute the following images.
 - $f(\mathbb{Z})$
 - $f(\mathbb{N})$
 - $f(\mathbb{R})$
- Let f be the function from \mathbb{R} to \mathbb{R} defined by $f(x) = x^2$. Use set-builder notation to describe the following preimages.
 - $f^{-1}(\{1\})$
 - $f^{-1}(\{x \mid 0 < x < 1\})$
 - $f^{-1}(\{x \mid x > 7\})$

Important definitions: Let $f : A \rightarrow B$.

- Domain and codomain:** the domain is A and the codomain is B .
- Image:** the image of f is $f(A) = \{b \in B \mid f(a) = b \text{ for some } a \in A\}$.
- Well-defined:** (i) $f(A) \subseteq B$, and (ii) if $f(a) = b$ and $f(a) = b'$, then $b = b'$.
(Example: $f(x) = \sqrt{x}$ with codomain $\mathbb{R}_{\geq 0}$. Non-example: $f(x) = \sqrt{x}$ with codomain \mathbb{R} .)
- Preimage:** the preimage of $b \in B$ is $f^{-1}(b) = \{a \in A \mid f(a) = b\}$.
- Invertible:** For all $b \in B$, $f^{-1}(b)$ has exactly one element.
- Injective:** if $f(a) = f(a')$, then $a = a'$.
(Example: $f(x) = x^2$ on domain $\mathbb{R}_{\geq 0}$. Non-example: $f(x) = x^2$ on domain \mathbb{R} .)
- Surjective:** for all $b \in B$ in the codomain, there is some $a \in A$ such that $f(a) = b$ (i.e. $f(A) = B$).
(Example: $f(x) = x^2$ with codomain $\mathbb{R}_{\geq 0}$. Non-example: $f(x) = x^2$ with codomain \mathbb{R} .)
- Bijective:** surjective and injective.

Functions

Some functions you might be familiar with:

$$f(x) = x^2, \quad f(x) = 3x-2, \quad f(x) = \sqrt{x}, \quad f(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}.$$

A couple more we'll need:

- ▶ For $x \in \mathbb{R}$, the **floor** of x is the greatest integer that is less than or equal to x , written $\lfloor x \rfloor$. For example,

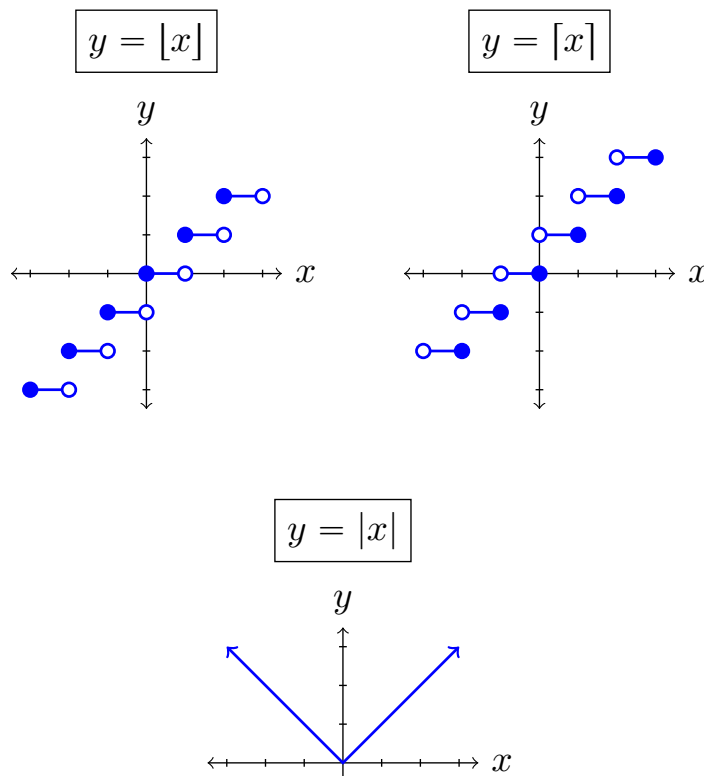
$$\lfloor 1/2 \rfloor = 0, \quad \lfloor -1/2 \rfloor = -1, \quad \lfloor 13 \rfloor = 13, \quad \lfloor \pi \rfloor = 3.$$

- ▶ For $x \in \mathbb{R}$, the **ceiling** of x is the least integer that is greater than or equal to x , written $\lceil x \rceil$. For example,

$$\lceil 1/2 \rceil = 1, \quad \lceil -1/2 \rceil = 0, \quad \lceil 13 \rceil = 13, \quad \lceil \pi \rceil = 4.$$

- ▶ The **absolute value** of a real number x is

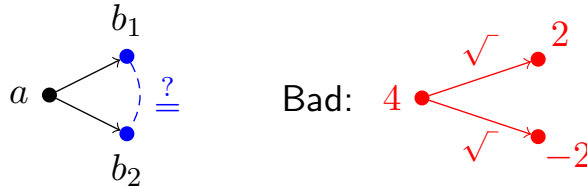
$$|x| = \begin{cases} x & \text{if } x \text{ is nonnegative,} \\ -x & \text{if } x \text{ is negative,} \end{cases}$$



What makes a function?

- ▶ You need a **domain** (input).
- ▶ The function should be **well-defined** (part 1): for every input, there is exactly one output. Namely,

$$\boxed{\text{if } f(a) = b_1 \text{ and } f(a) = b_2, \text{ then } b_1 = b_2.}$$



The domain together with a function determines a **range** or **image** (output).

Example

Consider $f(x) = x^2$.

If the domain is \mathbb{R} , then the range is $\mathbb{R}_{\geq 0}$.

If the domain is $\{-1\}$, then the range is $\{1\}$.

Either way, f is well-defined “on its domain”.

Like we can pick a universal set, we can also pick a **codomain**, a set containing the range of f .

If f is a function with domain A and codomain B , we say f is a **function** or **map** or **transformation** from A to B , and we write

$$f : A \rightarrow B.$$

For $a \in A$, we write

$$f : a \mapsto f(a),$$

where “ \mapsto ” reads “maps to”.

If you have a function $f : A \rightarrow B$, and $A' \subseteq A$, you can **restrict** f to the domain A' , written

$$f|_{A'} : A' \rightarrow B.$$

This means that the definition of the function doesn't change, you just consider its image on fewer elements.

If you pick a bad codomain, your expression is no longer a function (not well-defined, part 2).

Example

$$f : \mathbb{R} \rightarrow \mathbb{Z} \text{ defined by } x \mapsto x$$

is not a function.

Example

Consider the function

$$f : \mathbb{R} \rightarrow \mathbb{R} \\ x \mapsto x^2.$$

Then the image of f is $\mathbb{R}_{\geq 0}$. If we restrict f to $\{-1\} \subseteq \mathbb{R}$, the image of $f|_{\{-1\}} : \{-1\} \rightarrow \mathbb{R}$ is $\{1\}$.

The functions

$$g : \mathbb{R} \rightarrow \mathbb{C} \quad \text{and} \quad h : \mathbb{R} \rightarrow \mathbb{C} \cup \mathbb{C}^{15} \\ x \mapsto x^2 \quad \quad \quad x \mapsto x^2$$

both have image $\mathbb{R}_{\geq 0}$.

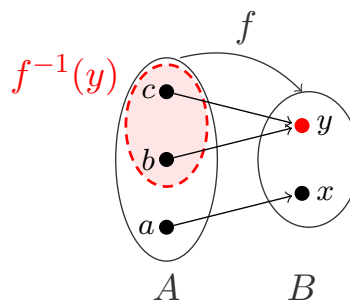
The map

$$\varphi : \mathbb{R} \rightarrow \mathbb{Z} \\ x \mapsto x^2$$

is not well-defined, since the image is not contained in the codomain.

The **image of an element** $a \in A$ is just $f(a)$. The preimage is defined on any element of subset of the codomain. Namely, the **preimage** of $b \in B$ is the set of elements $a \in A$ such that $f(a) = b$:

$$f^{-1}(b) = \{a \in A \mid f(a) = b\}.$$



The **preimage of a subset** $B' \subseteq B$ is defined similarly, only using containment:

$$f^{-1}(B') = \{a \in A \mid f(a) \in B'\}.$$

Notice, either way, a preimage is a **set**!!

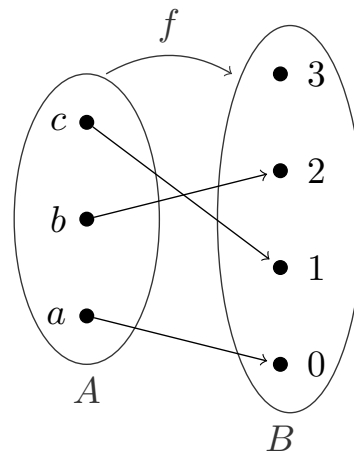
A function $f : A \rightarrow B$ is **invertible** if for every $b \in B$, $f^{-1}(b)$ has exactly one element.

A function is called **one-to-one** or **injective** if every element in the range has at most one element in its preimage.

Some examples of **injective functions**:

$$f(x) = 3x - 5 \text{ with domain } \mathbb{C}, \quad f(x) = x^2 \text{ with domain } \mathbb{R}_{\geq 0},$$

$$f(x) = \lfloor x \rfloor \text{ with domain } \mathbb{Z},$$



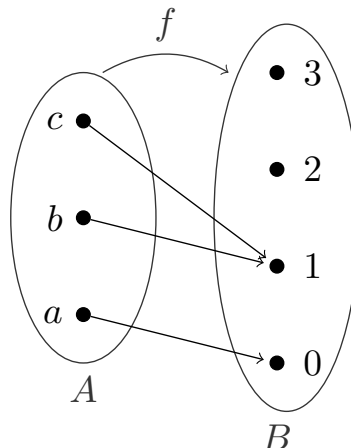
A function is called **one-to-one** or **injective** if every element in the range has at most one element in its preimage.

Some examples of **functions that are not injective**:

$$f(x) = 3x - 5 \text{ with domain time on a clock,}$$

$$f(x) = x^2 \text{ with domain } \mathbb{R},$$

$$f(x) = \lfloor x \rfloor \text{ with domain } \mathbb{Q},$$



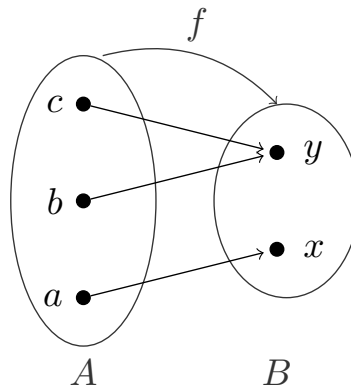
A function is called *onto* or *surjective* if the codomain and the image are the same thing.

Some examples of **surjective functions**:

$$f(x) = 3x - 5 \text{ with domain and codomain } \mathbb{C},$$

$$f(x) = x^2 \text{ with domain } \mathbb{R} \text{ and codomain } \mathbb{R}_{\geq 0},$$

$$f(x) = \lfloor x \rfloor \text{ with domain } \mathbb{R} \text{ and codomain } \mathbb{Z},$$



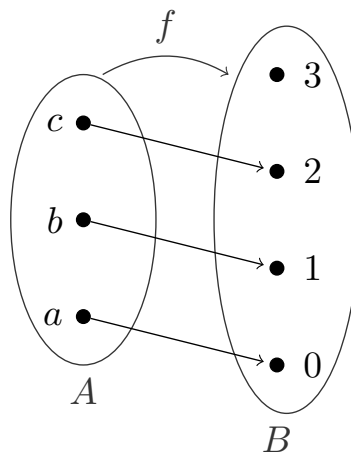
A function is called *onto* or *surjective* if the codomain and the image are the same thing.

Some examples of **functions that are not surjective**:

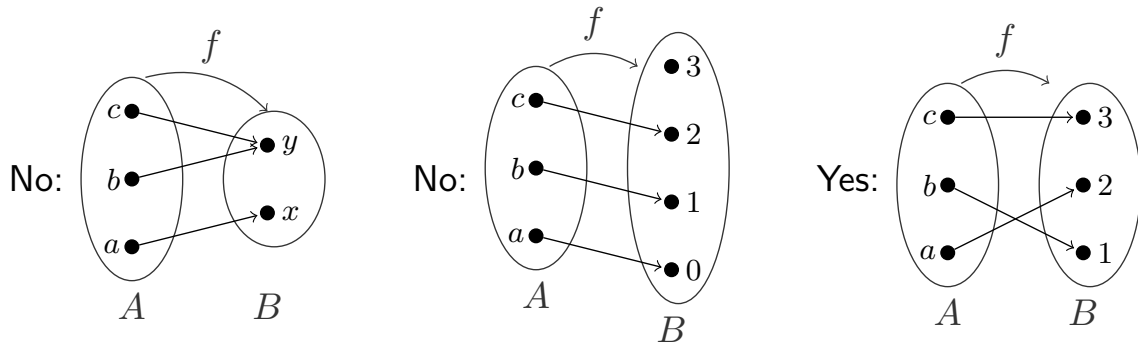
$$f(x) = 3x - 5 \text{ with domain } \mathbb{R} \text{ and codomain } \mathbb{C},$$

$$f(x) = x^2 \text{ with domain and codomain } \mathbb{R},$$

$$f(x) = \lfloor x \rfloor \text{ with domain and codomain } \mathbb{Q},$$



A function that is both injective and surjective is **bijective** or a **one-to-one correspondence**.



Theorem

A function $f : A \rightarrow B$ is bijective if and only if it is invertible.

You try: Exercise 8.

Let

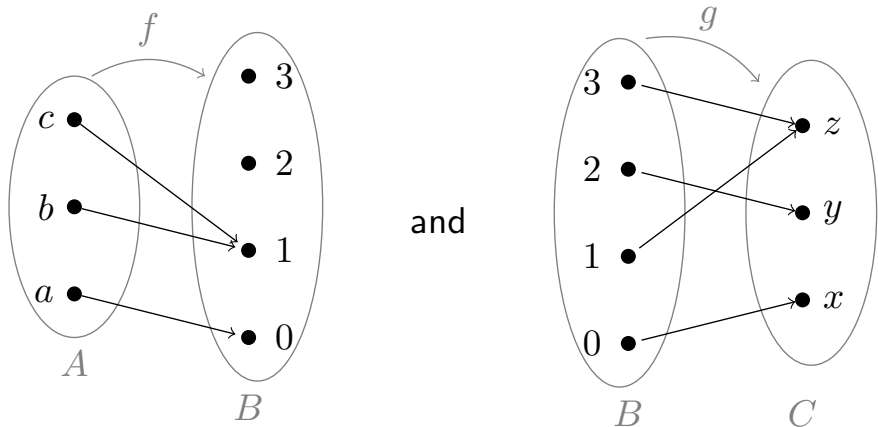
$$f : A \rightarrow B \quad \text{and} \quad g : B \rightarrow C.$$

Then the **composition** of g and f is

$$g \circ f = g(f(a)) : A \rightarrow C.$$

Example

Let



What is $g \circ f$?

Let

$$f : A \rightarrow B \quad \text{and} \quad g : B \rightarrow C.$$

Then the **composition** of g and f is

$$g \circ f = g(f(a)) : A \rightarrow C.$$

Example

Let $f(x) = x^2 + 1$ and let $g(x) = \lfloor x \rfloor$, both with domain and codomain \mathbb{R} . Since the domain and codomain are equal for both, I can consider both $f \circ g$ and $g \circ f$. We have

$$f \circ g = \lfloor x \rfloor^2 + 1 \quad \text{and} \quad g \circ f = \lfloor x^2 + 1 \rfloor.$$

You try: Exercise 9.

Theorem

Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be functions. If both f and g are one-to-one functions, then $g \circ f$ is also one-to-one.

You try: Exercise 10.