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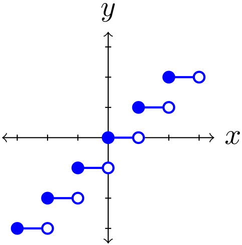

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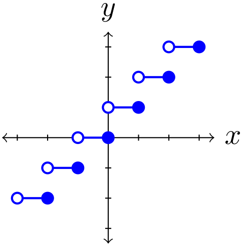

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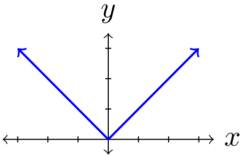

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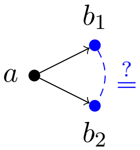

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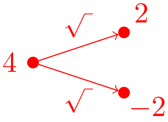

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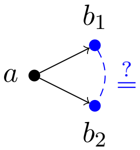

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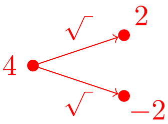

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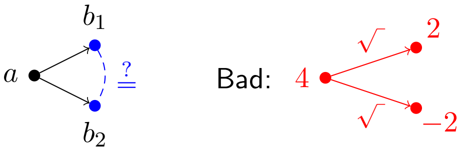

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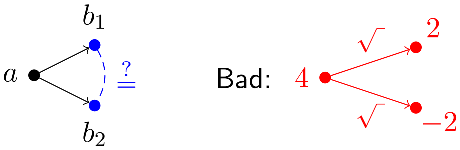

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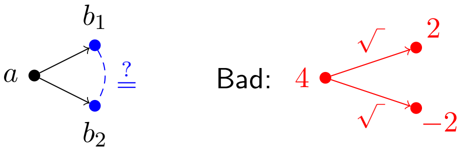

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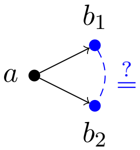
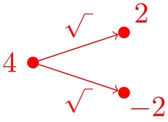


Bad: 

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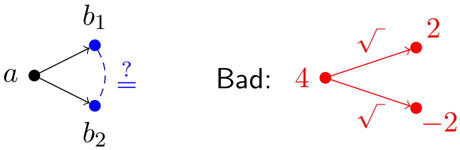

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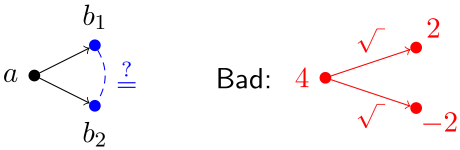

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Either way, f is well-defined "on its domain".

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$$f : \mathbb{R} \to \mathbb{Z} \text{ defined by } x \mapsto x$$

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The map

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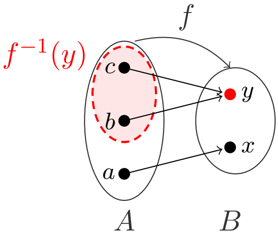

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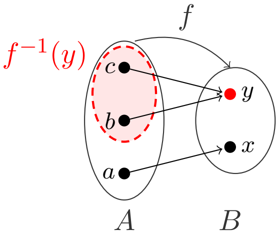

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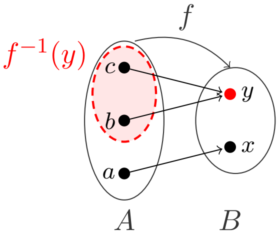

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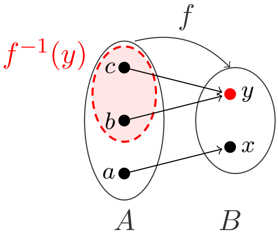

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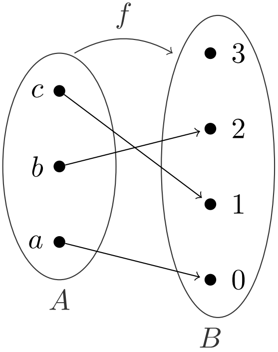

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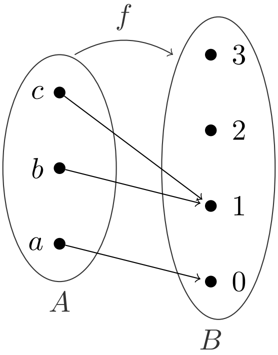

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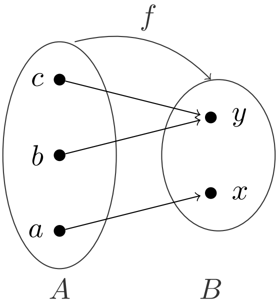

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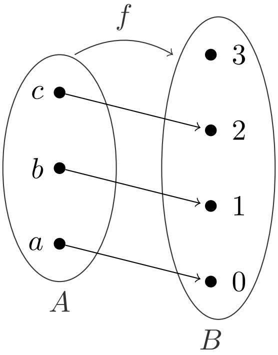

A function that is both injective and surjective is bijective or a one-to-one correspondence.

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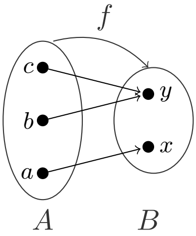
No:



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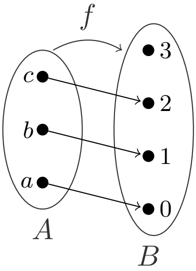
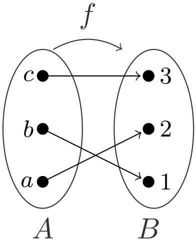


Yes:

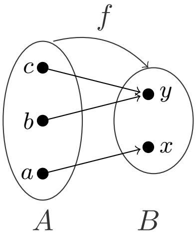


Theorem

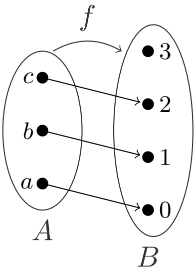
A function $f : A \to B$ is bijective if and only if it is invertible.

A function that is both injective and surjective is bijective or a one-to-one correspondence.

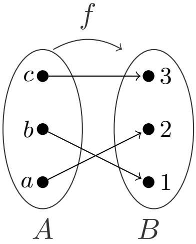
No:



No:



Yes:



Theorem

A function $f : A \to B$ is bijective if and only if it is invertible.

You try: Exercise 8.

Let

$$f : A \to B \qquad \text{and} \qquad g : B \to C.$$

Then the composition of g and f is

$$g \circ f = g(f(a)) : A \to C.$$

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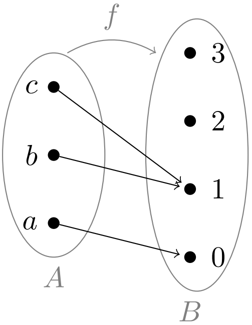
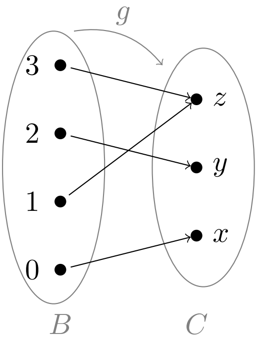
Example

Let



and



.

What is $g \circ f$?

Let

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Then the composition of g and f is

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Example

Let $f(x) = x^2 + 1$ and let $g(x) = \lfloor x \rfloor$, both with domain and codomain $\mathbb{R}$.

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Example

Let $f(x) = x^2 + 1$ and let $g(x) = \lfloor x \rfloor$, both with domain and codomain $\mathbb{R}$. Since the domain and codomain are equal for both, I can consider both $f \circ g$ and $g \circ f$.

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$$f \circ g = \lfloor x \rfloor^2 + 1 \qquad \text{and} \qquad g \circ f = \lfloor x^2 + 1 \rfloor.$$

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You try: Exercise 9.

Let

$$f : A \to B \qquad \text{and} \qquad g : B \to C.$$

Then the composition of g and f is

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You try: Exercise 9.

Theorem

Let $f : A \to B$ and $g : B \to C$ be functions. If both f and g are one-to-one functions, then $g \circ f$ is also one-to-one.

Let

$$f : A \to B \qquad \text{and} \qquad g : B \to C.$$

Then the composition of g and f is

$$g \circ f = g(f(a)) : A \to C.$$

Example

Let $f(x) = x^2 + 1$ and let $g(x) = \lfloor x \rfloor$, both with domain and codomain $\mathbb{R}$. Since the domain and codomain are equal for both, I can consider both $f \circ g$ and $g \circ f$. We have

$$f \circ g = \lfloor x \rfloor^2 + 1 \qquad \text{and} \qquad g \circ f = \lfloor x^2 + 1 \rfloor.$$

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Theorem

Let $f : A \to B$ and $g : B \to C$ be functions. If both f and g are one-to-one functions, then $g \circ f$ is also one-to-one.

You try: Exercise 10.