Math 365 – Wednesday 1/30/19

Course website: https://zdaugherty.ccnysites.cuny.edu/teaching/m365s19/

Warmup. Recall that the power set of a set A is

$$\mathcal{P} = \{X \mid X \subseteq A\}.$$

(1) What is $|\emptyset|$? What is $|\{\emptyset\}|$?
(2) Let $A = \{x\}$. Calculate $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.
(3) Let $A = \emptyset$. Calculate $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.
(4) Give an example of a set A such that $A \cap \mathcal{P}(A) = \emptyset$.
(5) Give an example of a set A such that $A \cap \mathcal{P}(A) \neq \emptyset$.
(6) True or false and why: For any set A, $\{\emptyset\} \subseteq \mathcal{P}(\mathcal{P}(A))$.
(7) Explain why $\mathcal{P}(A) \cap \mathcal{P}(\mathcal{P}(A)) \neq \emptyset$.

Exercise 3. Fix two sets A and B and a universal set U. Draw and shade in diagrams to decide whether the following identities are true or false. For those that are false, give a concrete example illustrating the failure of the identity. (Your argument against should start with something like *"Let $A =$??? and $B =$???. Then…"*.)

(a) $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$
(b) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$
(c) $A \cap (A \cup B) = (U - (B - A)) - \overline{(A \cup B)}$

Exercise 4. Let $A_i = \{1, 2, \dots, i\}$ for $i = 1, 2, 3, \dots$.
Calculate

$$\bigcup_{i=1}^{n} A_i \qquad \text{and} \qquad \bigcap_{i=1}^{n} A_i$$

for $n = 2, 4, 5$ (meaning compute $\bigcup_{i=1}^{2} A_i$, $\bigcup_{i=1}^{3} A_i$, etc.). Make a hypothesis about what $\bigcup_{i=1}^{n} A_i$ and $\bigcap_{i=1}^{n} A_i$ are for general n. Then make a hypothesis about what $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ are. Explain in words why. (More formal proof comes next.)

Exercise 5. Show that if $A_i = \{1, 2, \dots, i\}$ for $i = 1, 2, 3, \dots$, then $\bigcap_{i=1}^{\infty} A_i = \{1\}$.
[Hint: Call $\mathcal{A} = \bigcap_{i=1}^{\infty} A_i$ for brevity. First argue that $\{1\} \subseteq \mathcal{A}$. For the reverse, explain why $\mathcal{A} = A_1 \cap \bigcap_{i=2}^{\infty} A_i$, and use that expression to show that $\mathcal{A} \subseteq \{1\}$. You will want to use the fact that the $X \cap Y$ of two sets X and Y is a subset of X (also it's a subset of Y)]

Exercise 6. Argue formally that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Exercise 7. Find

$$\bigcup_{i=1}^{\infty} A_i \qquad \text{and} \qquad \bigcap_{i=1}^{\infty} A_i$$

for the following. In each case, justify your answer with a formal argument (like in Exercise 5).

(a) $A_i = \{i, i+1, i+2, \dots\}$
(b) $A_i = \{0, i\}$
(c) $A_i = (i, \infty) = \{x \in \mathbb{R} \mid i < x\}$

Warm up

Recall that the power set of a set A is

$$\mathcal{P} = \{X \mid X \subseteq A\}$$

.

1. What is $|\varnothing|$? What is $|\{\varnothing\}|$?
2. Let $A = \{x\}$. Calculate $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.
3. Let $A = \varnothing$. Calculate $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A))$.
4. Give an example of a set A such that $A \cap \mathcal{P}(A) = \varnothing$.
5. Give an example of a set A such that $A \cap \mathcal{P}(A) \neq \varnothing$.
6. True or false and why: For any set A, $\{\varnothing\} \subseteq \mathcal{P}(\mathcal{P}(A))$.
7. Explain why $\mathcal{P}(A) \cap \mathcal{P}(\mathcal{P}(A)) \neq \varnothing$.

Some shorthands you'll see in the book:

symbol:	means:	example:
$\in$	"in", "contained in"	"$x \in \mathbb{R}$" means "x is a real number".
$\forall$	"for all"	$A \subseteq B$ if $\forall a \in A$, we have $a \in B$.
$\wedge$	"and"	$A \cap B = \{x \in U \mid (x \in A) \wedge (x \in B)\}$.
$\vee$	"or" (inclusive)	$A \cup B = \{x \in U \mid (x \in A) \vee (x \in B)\}$.
$\neg$	"not"	$\overline{A} = \{x \in U \mid \neg(x \in A)\}$.

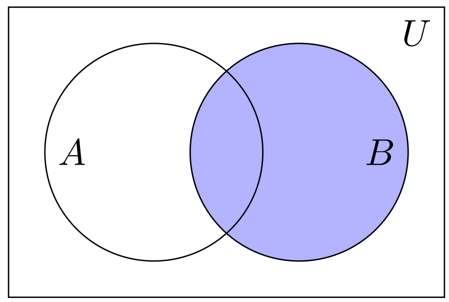
Put a priority on clarity!
Writing mathematics is not that different that any other writing. In journalism, clear and articulate writing is as important as content; the same is true in math. Don't make your reader work too hard to understand what you're trying to convey! In short, use symbols sparingly–go for clarity, not just saving space.

A Venn diagram is an abstract representation of a family of sets sitting inside of a universal set.

How to draw a Venn diagram:
Draw the universal set at a rectangle.
Inside that rectangle, indicate a set by drawing a closed loop (usually a circle, but not always) where the object in the set are the points inside that closed loop.

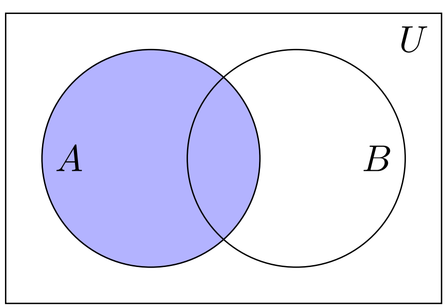
Say we have two sets A and B and a fixed universal set U:

B looks like

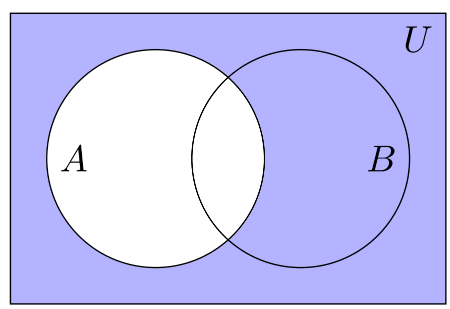


Shade in areas to indicate various sets.

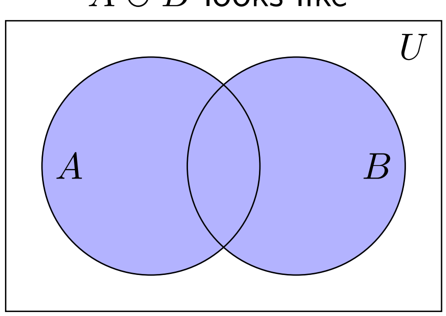
A looks like



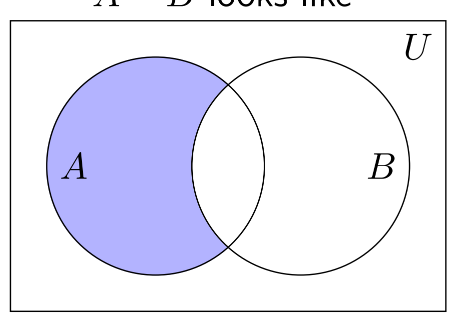
$\overline{A}$ looks like



$A \cup B$ looks like



$A - B$ looks like



You try: Do Exercise 3

Infinite unions and intersections

Recall summation and product notation

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n, \qquad \sum_{i=1}^{\infty} a_i = a_1 + a_2 + \cdots,$$

$$\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n, \qquad \prod_{i=1}^{\infty} a_i = a_1 \cdot a_2 \cdots,$$

for numbers $a_1, a_2, \ldots$.

Similarly, let $A_1, A_2, \ldots$ be a (possibly infinite) collection of sets.

For example,

$$A_1 = \{1\}, \quad A_2 = \{1, 2\}, \quad \ldots \qquad A_i = \{1, 2, 3, \ldots, i\}.$$

Then

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n, \qquad \bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n, \qquad \bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \cdots.$$

You try: Exercise 4.

How can we prove your hypothesis? For two sets A and B, we have

$$\boxed{A = B \quad \text{exactly when} \quad A \subseteq B \quad \text{and} \quad B \subseteq A.}$$

What does it *mean* for two sets to be equal?

Example: Compare

$$W = \{1,2\}, \quad X = \{1,2,3\}, \quad Y = \{1\}, \quad \text{and} \quad Z = \{1,2\}.$$

- $W \neq X$ because $3 \in X$ but $3 \notin W$ ($X \nsubseteq W$).
- $W \neq Y$ because $2 \in W$ but $2 \notin Y$ ($W \nsubseteq Y$).
- $W = Z$ because

  $1 \in W$ and $1 \in Z$;

  $2 \in W$ and $2 \in Z$;

  and there are no other elements in W or Z ($W \subseteq Z$ and $Z \subseteq W$).

In general, $A = B$ means that every element of A is in B, and vice versa. But

$A \subseteq B$ means that every element of A is in B,

(if $a \in A$, then $a \in B$ too)

and

$B \subseteq A$ means that every element of B is in A,

(if $b \in B$, then $b \in A$ too)

the "and vice versa" part.

For two sets A and B, we have

$$\boxed{A = B \quad \text{exactly when} \quad A \subseteq B \quad \text{and} \quad B \subseteq A.}$$

(If $a \in A$, then $a \in B$. And if $b \in B$, then $b \in A$.)

Example

If $A_i = \{1, 2, \ldots, i\}$ for $i = 1, 2, 3, \ldots$, then $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}_{>0}$.

Proof.

Let $\mathcal{A} = \bigcup_{i=1}^{\infty} A_i$.

First, if $a \in \mathcal{A}$, then a is an element of one of the A_i's. But since $A_i \subseteq \mathbb{Z}_{>0}$ for all $i = 1, 2, 3, \ldots$, we get that $a \in \mathbb{Z}_{>0}$. Thus $\mathcal{A} \subseteq \mathbb{Z}_{>0}$.

Next, if $i \in \mathbb{Z}_{>0}$, then i is an element of A_i, which is a subset of $\mathcal{A}$. Therefore i is and element of $\mathcal{A}$. So $\mathbb{Z}_{>0}$ is a subset of $\mathcal{A}$.

Therefore $\mathcal{A} = \mathbb{Z}_{>0}$. $\square$

You try: Do Exercise 5.

Let A, B, C be sets contained in a universal set U.
The following identities are our core set operations.

Identity	Name
$A \cap U = A \cup \varnothing = A$	Identity laws
$A \cup U = U$ and $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ (Exercise 6) $\overline{A \cap B} = \overline{A} \cup \overline{B}$	**De Morgan's laws**
$A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ and $A \cap \overline{A} = \varnothing$	Complement laws