Welcome to Math 345 – Monday 8/28/17

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Office hours: Mondays 12:45–1:45, or by appointment

Course Website: https://zdaugherty.ccnysites.cuny.edu/teaching/m345f17/

Homework 0: due Thursday 8/31 by email.
See course website for the rest of the syllabus, as well as instructions for homework 0. Follow these instructions precisely in your email to me to receive credit.

Attach at the end of Homework 1:
Before writing up homework 1, read handouts "Communicating Mathematics through Homework and Exams" and "Some Guidelines for Good Mathematical Writing". Then, at the end of your write-up, include the following, labeling this as **"Writing exercise"**.

(a) List three things you learned or thought about more carefully after reading these documents.

(b) Mark up this written homework assignment, showing where you followed or failed to follow the mechanical and stylistic issues outlined in *Communicating Mathematics....* How might you improve in the future?

(c) List three or more ways that you succeeded or failed at following the advice in *Some Guidelines....* How might you improve in the future?

To receive credit for this assignment, you must complete this exercise.

Types of numbers:

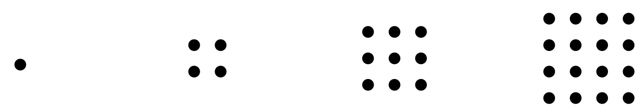
- *Integers*, or *whole numbers*, written $\mathbb{Z}$: $0, 1, -1, 2, -2, 3, -3, \dots$
- *Natural numbers*, written $\mathbb{Z}_{>0}$: $1, 2, 3, 4, \dots$
  In set notation,
  $$\mathbb{Z}_{>0} = \{n \in \mathbb{Z} \mid n > 0\}.$$
  "the set of" ($\{$)   "in" ($\in$)   "such that" ($\mid$)
- *Units*, numbers that have multiplicative inverses: (depends on what set you're working in!)
  (1) In $\mathbb{Z}$: 1 and -1.
  (2) In $\mathbb{R}$ (the real numbers): everything except for 0.
  (3) In $\mathbb{Z}_{>0}$: 1.
- *Prime numbers*, natural numbers p whose only divisors are 1 and p: $2, 3, 5, 7, 11, \dots$
  See last page for the list of primes under 3000.
- *Composite numbers*, natural numbers other than prime numbers and 1: $4, 6, 8, 9, 10, 12, \dots$

Exercise 1. When we take a composite number n and "factor it into primes", that means we write it as a product of prime numbers, usually in increasing order, using exponents to simplify:
$$n = p_1^{m_1} p_2^{m_2} \cdots p_\ell^{m_\ell}, \qquad p_1 < p_2 < \cdots < p_\ell \quad \text{primes.}$$
For example,
$$4840 = 2^3 \cdot 5 \cdot 11^2, \qquad 161 = 7 \cdot 23, \qquad 19683 = 3^9.$$
List the first 15 composite numbers, and factor them into primes.

Exercise 2. Square and triangular numbers

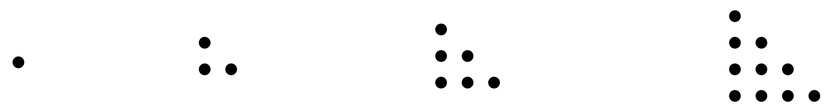
Square numbers are any number of the form n^2 for $n \in \mathbb{Z}_{>0}$:

$$1\ (=1^2) \qquad 4\ (=2^2) \qquad 9\ (=3^2) \qquad 16\ (=4^2)$$



Triangular numbers are any number of the form $1+2+3+\cdots+n$ for $n \in \mathbb{Z}_{>0}$:

$$1\ (=1) \qquad 3\ (=1+2) \qquad 6\ (=1+2+3) \qquad 10\ (=1+2+3+4)$$



(a) Draw pictures using dots to show that

$$\underbrace{3^2}_{\text{square}} = 2\underbrace{(1+2)}_{\text{triangular}} + 3 \text{ and } \quad \underbrace{4^2}_{\text{square}} = 2\underbrace{(1+2+3)}_{\text{triangular}} + 4.$$

(b) Draw a picture using dots to show that

$$\underbrace{(n+1)^2}_{\text{square}} = 2\underbrace{(1+2+\cdots+n)}_{\text{triangular}} + (n+1) \tag{1}$$

Deduce that

$$1+2+\cdots+n = n(n+1)/2 \tag{2}$$

(i.e. solve equation (1) for the triangular number).

(c) The number 1 is both square and triangular. Are there more?

(i) Do some examples by hand, and then try using a computer to generate more examples (generate the first 500 or so triangular numbers and their square roots using whatever your favorite computational program is—if you don't have one, try using a spreadsheet).

(ii) Consider equation (2): what can you say about the factors of n and $n+1$ that make it possible for $n(n+1)/2$ to be a perfect square?

(iii) Make a hypothesis: do you think there are finitely many or infinitely many numbers that are both triangular and square?

(d) Try adding up the first few odd numbers and see if the numbers you get satisfy some sort of pattern. Once you find the pattern, express it as a formula. Give a geometric verification (picture using dots) that your formula is correct. (Like in parts (a) and (b)).

Exercise 3. Twin primes.
The *twin primes* are the prime numbers p such that $p+2$ or $p-2$ is also a prime:

$$\boxed{3,5,7}, \boxed{11,13}, \ldots.$$

(a) List all twin primes under 1000 (see the table of primes under 3000 on the last page). What is the first odd prime that is not a twin prime?

(b) Why *don't* we have a name for a prime number p such that $p+1$ or $p-1$ is also a prime?

(c) Can you find any prime triplets, i.e. numbers p, $p+2$, and $p+4$ that are all prime? Are there are finitely many or infinitely many of these? (Look at your twin prime examples from part (a), and consider the odd numbers that come immediately before and after each pair of twin primes.)

(d) Do you think that there are infinitely many twin prime numbers? Do an internet search for "twin prime conjecture". Briefly, what is the current status of mathematic's progress on this question?

Exercise 4. Primes of the form. . . It is generally believed that there are infinitely many primes of the form n^2+1, though this is still an open question and work is being done all the time. For example, the first three are

$$5 = 2^2+1, \quad 17 = 4^2+1, \quad \text{and} \quad 37 = 6^2+1.$$

(a) By computing values of n^2+1, and observing which are prime, list the next 5 primes of this form. (You can discount odd values of n—why?)

(b) For what a do you think there might be infinitely many primes of the form n^2-a? To answer this, do the following:

(i) Generate data for

$$n^2-1, \quad n^2-2, \quad n^2-3, \quad \text{and} \quad n^2-4,$$

for $n = 1, 2, \ldots, 10$, factoring composite numbers into primes (use a computer to assist generating your data). Report your findings in a table of values:

n	n^2-1	n^2-2	n^2-3	n^2-4
2	3	2	1	0
3	$8=2^3$	7	$6=2\cdot 3$	5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(ii) For which values does n^2-a almost always factor?

(iii) Make a hypothesis.

Primes under 3000

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013
1019	1021	1031	1033	1039	1049	1051	1061	1063	1069
1087	1091	1093	1097	1103	1109	1117	1123	1129	1151
1153	1163	1171	1181	1187	1193	1201	1213	1217	1223
1229	1231	1237	1249	1259	1277	1279	1283	1289	1291
1297	1301	1303	1307	1319	1321	1327	1361	1367	1373
1381	1399	1409	1423	1427	1429	1433	1439	1447	1451
1453	1459	1471	1481	1483	1487	1489	1493	1499	1511
1523	1531	1543	1549	1553	1559	1567	1571	1579	1583
1597	1601	1607	1609	1613	1619	1621	1627	1637	1657
1663	1667	1669	1693	1697	1699	1709	1721	1723	1733
1741	1747	1753	1759	1777	1783	1787	1789	1801	1811
1823	1831	1847	1861	1867	1871	1873	1877	1879	1889
1901	1907	1913	1931	1933	1949	1951	1973	1979	1987
1993	1997	1999	2003	2011	2017	2027	2029	2039	2053
2063	2069	2081	2083	2087	2089	2099	2111	2113	2129
2131	2137	2141	2143	2153	2161	2179	2203	2207	2213
2221	2237	2239	2243	2251	2267	2269	2273	2281	2287
2293	2297	2309	2311	2333	2339	2341	2347	2351	2357
2371	2377	2381	2383	2389	2393	2399	2411	2417	2423
2437	2441	2447	2459	2467	2473	2477	2503	2521	2531
2539	2543	2549	2551	2557	2579	2591	2593	2609	2617
2621	2633	2647	2657	2659	2663	2671	2677	2683	2687
2689	2693	2699	2707	2711	2713	2719	2729	2731	2741
2749	2753	2767	2777	2789	2791	2797	2801	2803	2819
2833	2837	2843	2851	2857	2861	2879	2887	2897	2903
2909	2917	2927	2939	2953	2957	2963	2969	2971	2999