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Example: $a^i \pmod{6}$

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		2	3	4	5	6
	1	1	1	1	1	1
↑	2	4	2	4	2	4
a	3	3	3	3	3	3
↓	4	4	4	4	4	4
	5	1	5	1	5	1
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Example: $a^i \pmod{8}$

$\leftarrow i \rightarrow$

$\uparrow a \downarrow$

	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1
2	4	0	0	0	0	0	0
3	1	3	1	3	1	3	1
4	0	0	0	0	0	0	0
5	1	5	1	5	1	5	1
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So what if $\gcd(a, n) = 1$, but n is not prime?

Are there solutions to $a^i \equiv 1 \pmod{n}$ when $\gcd(a, n) = 1$, but n is not prime?

Example: $a^i \pmod{6}$

$\leftarrow i \rightarrow$

		2	3	4	5	6
	1	**1**	**1**	**1**	**1**	**1**
↑	2	4	2	4	2	4
a	3	3	3	3	3	3
↓	4	4	4	4	4	4
	5	**1**	**5**	**1**	**5**	**1**
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$\leftarrow i \rightarrow$

$\uparrow a \downarrow$

	2	3	4	5	6	7	8
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**
2	4	0	0	0	0	0	0
3	**1**	**3**	**1**	**3**	**1**	**3**	**1**
4	0	0	0	0	0	0	0
5	**1**	**5**	**1**	**5**	**1**	**5**	**1**
6	4	0	0	0	0	0	0
7	**1**	**7**	**1**	**7**	**1**	**7**	**1**
8	0	0	0	0	0	0	0

Are there solutions to $a^i \equiv 1 \pmod{n}$ when $\gcd(a, n) = 1$, but n is not prime?

Example: $a^i \pmod{10}$

$\leftarrow i \rightarrow$

$\uparrow a \downarrow$

	2	3	4	5	6	7	8	9	10
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**
2	4	8	6	2	4	8	6	2	4
3	**9**	**7**	**1**	**3**	**9**	**7**	**1**	**3**	**9**
4	6	4	6	4	6	4	6	4	6
5	5	5	5	5	5	5	5	5	5
6	6	6	6	6	6	6	6	6	6
7	**9**	**3**	**1**	**7**	**9**	**3**	**1**	**7**	**9**
8	4	2	6	8	4	2	6	8	4
9	**1**	**9**	**1**	**9**	**1**	**9**	**1**	**9**	**1**
10	0	0	0	0	0	0	0	0	0

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n	$\{1 \leqslant a \leqslant n-1 \mid \gcd(a,n) = 1\}$
2	$\{1\}$
3	$\{1,2\}$
4	$\{1,3\}$
5	$\{1,2,3,4\}$
6	$\{1,5\}$
7	$\{1,2,3,4,5,6\}$
8	$\{1,3,5,7\}$

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mod 4:

×	1	3
1	1	3
3	3	1

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1	1	3
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mod 6:

×	1	5
1	1	5
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×	1	5
1	1	5
5	5	1

mod 8:

×	1	3	5	7
1	1	3	5	7
3	3	1	7	5
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mod 8:

×	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

You try: Compute the integers $1 \leqslant a \leqslant 11$ that are relatively prime to 10, and compute their multiplication table modulo 10.

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Lemma

Let $\Phi(n) = \{x_1, x_2, \ldots, x_m\}$ be the set of numbers between 1 and $n-1$ that are relatively prime to n. Then, for any integer a with $\gcd(a, n) = 1$, the numbers

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Proof: Suppose $x_ka \equiv x_\ell a \pmod{n}$. Since $\gcd(a,n) = 1$, we can cancel the a's.

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Analog for composite modulus:

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Big question:

Are there solutions to $a^i \equiv 1 \pmod{n}$ when $\gcd(a,n) = 1$?

Step 3: Since $(p-1)!$ is relatively prime to p, we can cancel.

Analog for composite modulus:

Let $a \in \mathbb{Z}$ with $\gcd(a,n) = 1$, and let $\{x_1, x_2, \ldots, x_m\}$ be the set of numbers between 1 and $n-1$ relatively prime to n. So

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What is this value?

Euler's phi function

Let

$$\Phi(n) = \{ \text{ integers } 1 \leqslant x \leqslant n-1 \text{ relatively prime to } n \},$$

and define $\phi(n) = |\Phi(n)|$.

Examples:

n	$\{1 \leqslant a \leqslant n-1 \mid \gcd(a,n) = 1\}$	$\phi(n)$
2	$\{1\}$	1
3	$\{1,2\}$	2
4	$\{1,3\}$	2
5	$\{1,2,3,4\}$	4
6	$\{1,5\}$	2
7	$\{1,2,3,4,5,6\}$	6
8	$\{1,3,5,7\}$	4

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From your example: What is $\phi(10)$?

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Next time: $\phi(mn) = \phi(m)\phi(n)$ whenever $\gcd(m,n) = 1$.