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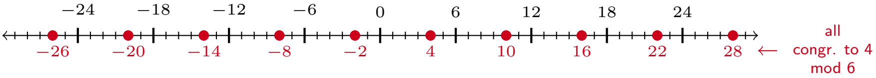

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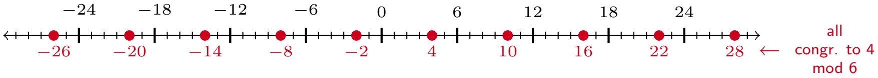

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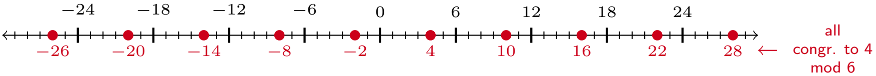

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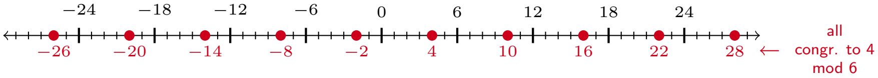

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$$12,\ 22,\ 31,\ \ldots, \quad \text{as well as} \quad -8,\ -18,\ -28,\ \ldots.$$

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If $\gcd(a, n) = 1$ and $ax \equiv b \pmod{n}$, then

1. compute $1 \leqslant k < n$ such that $ka \equiv 1 \pmod{n}$, so that
2. $x \equiv kb \pmod{n}$.

You try: Compute x such that

(1) $3x \equiv 7 \pmod{10}$ (2) $5x \equiv 2 \pmod{9}$

and check your answer.

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All solutions: Find one solution $u, v \in \mathbb{Z}$ to $d = ua + vn$. If $d|b$, then the solutions to $ax \equiv b \pmod{n}$ are given by

$$x = u(b/d) + \ell(n/d), \qquad \text{for} \quad \ell = 0, 1, \ldots, d-1.$$

To summarize

We've solved congruences of the form

$$ax \equiv b \pmod{n}.$$

Namely, we have two cases: Calculate $d = \gcd(a, n)$.

1. If $d \nmid b$, then there are no solutions.
2. If $d | b$, then there are exactly d solutions (mod n). Find them as follows:
   (a) Find one solution, either by guessing...

   If $d = 1$ and you can find an a' satisfying $a'a \equiv 1 \pmod{n}$, then
   $$x \equiv_n (a'a)x \equiv_n a'(ax) \equiv_n a'b.$$

   ...or by using the Euclidean algorithm to calculate

   $$ua + vn = d, \quad \text{so that} \quad b = (b/d)d = (b/d)ua + (b/d)vn.$$

   Thus $x = (b/d)u$ is one solution.

   (b) For the rest, add n/d until you have a full set.

Nonlinear congruences

Theorem (Polynomial Roots Mod p Theorem)

Let p be prime in $\mathbb{Z}_{>0}$, and let

$$f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x],$$

with $n \geqslant 1$ and $p \nmid a_n$. Then the congruence

$$f(x) \equiv 0 \pmod{n}$$

has at most n incongruent solutions.