Warmup

Recall that a "proof by induction" is done as follows: for a statement $S(n)$ that depends on an integer n,

1. prove a base (smallest) case; and
2. show that if $S(n)$ is true (the "induction hypothesis"), then so is $S(n+1)$.

You try: Prove the following identities using proof by induction.

(a) $1 + 2 + 3 + \cdots + n = n(n+1)2$

(b) For $a \neq 1$,

$$1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}.$$

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Strong induction: The inductive hypothesis becomes "assume $S(m)$ is true for *all* (base case)$\leqslant m \leqslant n$"; then the inductive step is to show $S(n+1)$ is true using any of those $S(m)$ for smaller m.

Last time:

Let $m, n \in \mathbb{Z}$ with $m \neq 0$. We say that m divides n if

$$n = mk \quad \text{for some} \quad k \in \mathbb{Z}, \quad \text{written } m|n.$$

The greatest common divisor of $a, b \in \mathbb{Z}_{>0}$, denoted $\gcd(a, b)$ is largest number that divides both a and b.

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$$\gcd(a,b)|a \text{ and } b, \qquad \text{and} \qquad \text{if } d|a \text{ and } b, \text{ then } d|\gcd(a,b).$$

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Theorem

Let a and b be nonzero integers, and let $g = \gcd(a, b)$.

(1) *For all $x, y \in \mathbb{Z}$, we have $g|(ax + by)$.*

(2) *The equation $ax_0 + by_0 = \gcd(a, b)$ always has at least one integer solution, which can be found via the Euclidean algorithm.*

(3) *All integers solutions to $ax + by = \gcd(a, b)$ are given by*

$$x = x_0 + \frac{kb}{\gcd(a,b)} \quad \text{and} \quad y = y_0 - \frac{ka}{\gcd(a,b)}, \quad k \in \mathbb{Z}.$$

Primes and their properties

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Theorem (Prime Divisibility Property)

Let p be a prime number, and suppose that p divides the product $a_1 a_2 \cdots a_r$, where $a_i \in \mathbb{Z}$. Then p divides at least one of the factors a_1, a_2, . . . , a_r.

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Let's look at examples where "unique factorization into primes" fails. . .

Even numbers

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But now, notice: $6, 8, 10$, and 30 are all prime in $2\mathbb{Z}_{>0}$, but

$$6 * 30 = 180 = 8 * 10.$$

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But now, notice: ± 2 and $1 \pm \sqrt{5}$ are all prime in $\mathbb{Z}[\sqrt{5}]$, but

$$2(-2) = -4 = (1 + \sqrt{5})(1 - \sqrt{5}).$$

Back to positive integers. . .

Theorem (The Fundamental Theorem of Arithmetic)

Every integer $n \geqslant 2$ *can be factored uniquely as*

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Congruences

Recall the division algorithm says for any $a, n \in \mathbb{Z}$ with $n \neq 0$, there are unique integers q and r satisfying

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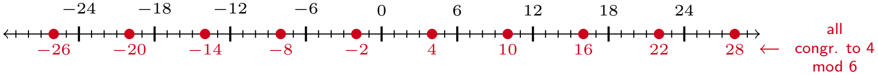

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$$a = q_a n + r_a \quad \text{and} \quad b = q_b n + r_b, \quad 0 \leqslant r_a, r_b < n,$$

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Congruences

For integers a, b, n, with $n \neq 0$, we say a is congruent to b modulo (mod) n, written

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Congruences

Theorem

For integers a, b, n, with $n \neq 0$, the following are equivalent:

1. *a and b have the same remainders when divided by n;*
2. *n divides $a - b$.*

Either way, we say that a is congruent to b modulo (mod) n, written

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1. Congruent is an equivalence relation (reflexive, symmetric, transitive).
2. If $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n}$, then
   (a) $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$, and
   (b) $a_1 a_2 \equiv b_1 b_2 \pmod{n}$.