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Let $m, n \in \mathbb{Z}$ with $m \neq 0$. We say that m divides n if n is a multiple of m, i.e.

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Example: the divisors of 28 are 1, 2, 4, 7, 14, and 28.

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We calculate $\gcd(a, b)$ either by comparing the prime factorizations (for small a, b) or by using the Euclidean algorithm.

Euclidean algorithm

The division algorithm says for any $a, b \in \mathbb{Z}$ with $b \neq 0$, there are unique integers q and r satisfying

$$a = bq + r \qquad \text{and} \qquad 0 \leqslant r < |b|.$$

Think: "a divided by b is q with remainder r."

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Think: "a divided by b is q with remainder r."
Repeatedly apply the division algorithm to find the GCD:

$$\begin{array}{ccccccl}
a & = & b * q_1 & + & r_1 & & \\
b & = & r_1 * q_2 & + & r_2 & & \\
r_1 & = & r_2 * q_3 & + & r_3 & & \\
 & \vdots & & & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} & & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow & \gcd(a, b) \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow & r_n
\end{array}$$

On the homework, you show:

For any positive integers a and b, there exist integers x and y satisfying $\gcd(a, b) = ax + by$.

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$$\gcd(35,100) = 7 \quad \text{and} \quad (3)35 + (-1)100 = 5$$
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Strategy: Take the Euclidean algorithm and solve for r_{n-1}, starting from the end...

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r_{n-5} & = & r_{n-4} * q_{n-3} & + & r_{n-3} \\
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-3	-81	-72	-63	-54	-45	-36	-27	-18	-9	0	9
-2	-69	-60	-51	-42	-33	-24	-15	-6	3	12	21
-1	-57	-48	-39	-30	-21	-12	-3	6	15	24	33
0	-45	-36	-27	-18	-9	0	9	18	27	36	45
1	-33	-24	-15	-6	3	12	21	30	39	48	57
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-10	-210	-201	-192	-183	-174	-165	-156	-147	-138	-129	-120	-111	-102	-93	-84	-75	-66	-57	-48	-39	-30	-21
-9	-198	-189	-180	-171	-162	-153	-144	-135	-126	-117	-108	-99	-90	-81	-72	-63	-54	-45	-36	-27	-18	-9
-8	-186	-177	-168	-159	-150	-141	-132	-123	-114	-105	-96	-87	-78	-69	-60	-51	-42	-33	-24	-15	-6	3
-7	-174	-165	-156	-147	-138	-129	-120	-111	-102	-93	-84	-75	-66	-57	-48	-39	-30	-21	-12	-3	6	15
-6	-162	-153	-144	-135	-126	-117	-108	-99	-90	-81	-72	-63	-54	-45	-36	-27	-18	-9	0	9	18	27
-5	-150	-141	-132	-123	-114	-105	-96	-87	-78	-69	-60	-51	-42	-33	-24	-15	-6	3	12	21	30	39
-4	-138	-129	-120	-111	-102	-93	-84	-75	-66	-57	-48	-39	-30	-21	-12	-3	6	15	24	33	42	51
-3	-126	-117	-108	-99	-90	-81	-72	-63	-54	-45	-36	-27	-18	-9	0	9	18	27	36	45	54	63
-2	-114	-105	-96	-87	-78	-69	-60	-51	-42	-33	-24	-15	-6	3	12	21	30	39	48	57	66	75
-1	-102	-93	-84	-75	-66	-57	-48	-39	-30	-21	-12	-3	6	15	24	33	42	51	60	69	78	87
0	-90	-81	-72	-63	-54	-45	-36	-27	-18	-9	0	9	18	27	36	45	54	63	72	81	90	99
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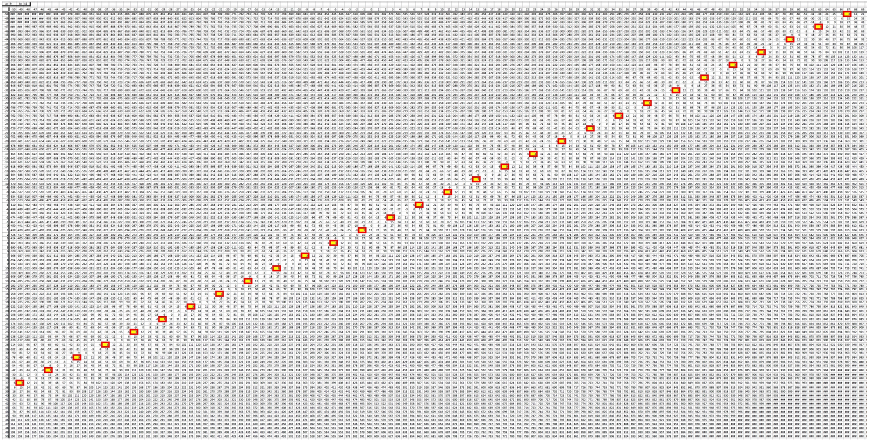

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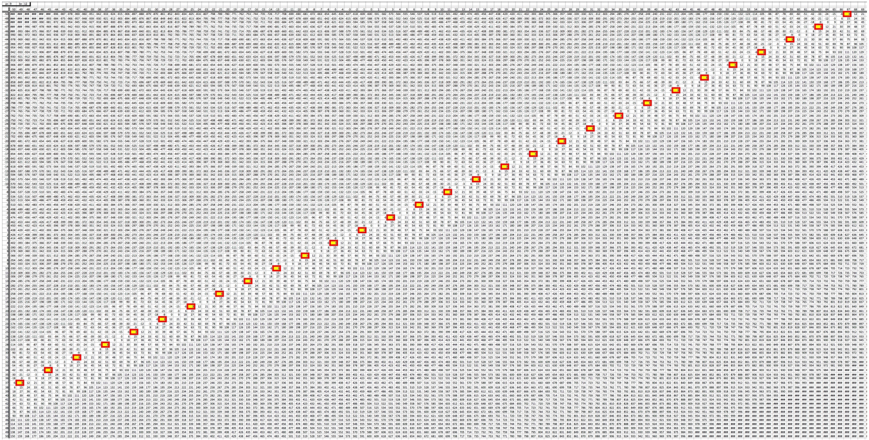

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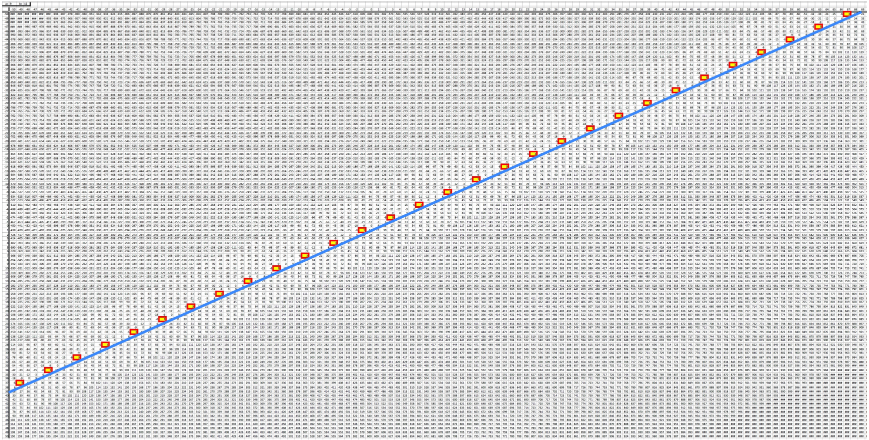

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bt and at are both integers (for the same t). (∗)

1. $t \in \mathbb{Q}$
2. If $t = n/m$ in lowest terms, then (∗) if and only if $m|a$ and $m|b$.

Lemma. For $a, b, x, y \in \mathbb{Z}$,

$$ax + by = a(x + bt) + b(y - at), \quad \text{for any } t \in \mathbb{Q}.$$

[Note: this is true for any t... real, complex, indeterminate, whatever. But the only hope that we have that $x + bt$ and $y - at$ could be integers is if t is at least rational.]

Now, given some $x, y \in \mathbb{Z}$ satisfying

$$ax + by = \gcd(a, b),$$

how do we generate more *integer solutions* x' and y' to $ax' + by' = \gcd(a, b)$? Namely,

when are $x + bt$ and $y - at$ both integers (for the same t)?

This happens exactly whenever

$$bt \text{ and } at \text{ are both integers (for the same } t). \qquad (*)$$

1. $t \in \mathbb{Q}$
2. If $t = n/m$ in lowest terms, then $(*)$ if and only if $m|a$ and $m|b$.

So $\boxed{t = k/\gcd(a, b)}$ for any $k \in \mathbb{Z}$ works!

Theorem

Let a and b be nonzero integers, and let $g = \gcd(a, b)$.

(1) *If $ax + by = z$ for $x, y \in \mathbb{Z}$, then $g|z$. (homework)*

(2) *The equation $ax_1 + by_1 = g$ always has at least one integer solution, which can be found via the Euclidean algorithm.*

(3) *The integers solutions to $g = ax + by$ are given by*

$$x = x_1 + \frac{kb}{g} \quad \text{and} \quad y = y_1 - \frac{ka}{g}, \quad k \in \mathbb{Z}.$$