Lecture 4: Divisibility and Greatest Common Divisor

Divisors

Let $m, n \in \mathbb{Z}$ with $m \neq 0$. We say that m divides n if n is a multiple of m, i.e.

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We often restrict to talking about numbers $n \in \mathbb{Z}_{>0}$, and list the divisors as the positive integers that divide n.

Example: the divisors of 12 are 1, 2, 3, 4, 6, and 12.

Common divisors

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If $b|a$, then $\gcd(a, b) = b$.
If $\gcd(a, b) = 1$, we say that a and b are relatively prime.

Example:

The divisors of 25 are 1, 5, and 25;
the divisors of 12 are 1, 2, 3, 4, 6, and 12;
so 25 and 12 are relatively prime (even though neither is prime).

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$$19500 = 2^2 * 3 * 5^3 * 13 \quad \text{and} \quad 440 = 2^3 * 5 * 11,$$

$$\text{so} \quad \gcd(19500, 400) = 2^2 * 5 = \boxed{20}.$$

In other words, $\gcd(a, b)$ will be the product over primes p to the highest power n such that $p^n|a$ and $p^n|b$.

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You try: compute the prime factorizations of $12, 30, 35$, and 84, and use them to compute

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Not computationally efficient either! (Prime factorization is computationally difficult/not possible without a list of primes.)

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$$a = bq + r \qquad \text{and} \qquad 0 \leqslant r < |b|.$$

Think: "a divided by b is q with remainder r."

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Ex: if $a = 17, b = 5$, then $q = 3$ and $r = 2$ since $17 = 5 * 3 + 2$.

Ex: if $a = -17, b = 5$, then $q = -4$ and $r = 3$ since $-17 = 5 * (-4) + 2$.

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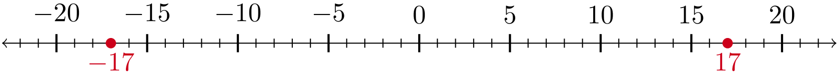

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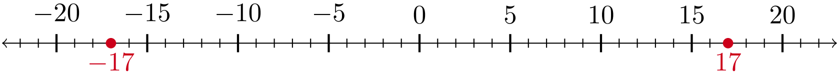

Proof: (sketch) If a and b are the same sign, subtract b from a until the result is between 0 and $|b| - 1$. The result is r and the number of subtractions is q. If they're different signs, add b to a until the result is between 0 and $|b| - 1$. The result is r and the number of additions is $-q$.

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if $a = 17, b = 5$, then $q = 3$ and $r = 2$ since $17 = 5 * 3 + 2$.

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Notice: $\gcd(17, 5) = 1$.

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Play this game again with new a and b:

1. Start with $a_1 = a$ and $b_1 = b$.
2. Compute the quotient q_i and remainder r_i in dividing a_i by b_i.
3. Repeat the division algorithm using $a_i = b_{i-1}$ and $b_i = r_{i-1}$.
4. Iterate until you get $r_n = 0$.
   Then compare $\gcd(a, b)$ with r_{n-1}.

You try: Do this process with $a = 30$, $b = 12$, and then with $a = 84$, $b = 30$.

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Claim: If n is the first time that $r_n = 0$, then $r_{n-1} = \gcd(a, b)$.

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You try: Do this process with $a = 30$, $b = 12$, and then with $a = 84$, $b = 30$.

Claim: If n is the first time that $r_n = 0$, then $r_{n-1} = \gcd(a, b)$.

Note that if $r = 0$ in the first step, then $b|n$, so $\gcd(a, b) = b$.

Spreadsheet functions

For a and integer and b a positive integer,

$$=\text{FLOOR}(a, b)$$

gives the largest multiple of b less or equal to a.

Namely, if $a = bq + r$, then $\text{FLOOR}(a, b) = bq$.

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Example:

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So to compute q and r such that $a = bq + r$,

$=$FLOOR$(a, b)/b$ returns q,

$=a-$FLOOR(a, b) returns r.

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\begin{array}{ccccc}
a & = & b * q_1 & + & r_1 \\
b & = & r_1 * q_2 & + & r_2 \\
r_1 & = & r_2 * q_3 & + & r_3 \\
 & \vdots & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} \quad \leftarrow \gcd(a, b)? \\
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\overset{r_0}{\cancel{b}} & = & r_1 * q_2 & + & r_2 & & \\
r_1 & = & r_2 * q_3 & + & r_3 & & \\
 & \vdots & & & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} & & \\
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\end{array}
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To make everything look the same, let $r_{-1} = a$ and $r_0 = b$. So every line comes in the form

$$r_{i-2} = r_{i-1} * q_i + r_i.$$

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Let $r_{-1} = a$ and $r_0 = b$, so that the algorithm looks like

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\end{array}$$

Last line: $r_{n-2} = r_{n-1} q_n$.

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$$\begin{array}{rcccc} r_{-1} & = & r_0 * q_1 & + & r_1 \\ r_0 & = & r_1 * q_2 & + & r_2 \\ r_1 & = & r_2 * q_3 & + & r_3 \\ & \vdots & & & \\ r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} \\ r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a,b)? \\ r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n \end{array}$$

Last line: $r_{n-2} = r_{n-1} q_n$.

So

$r_{n-3} = r_{n-2} q_{n-1} + r_{n-1}$

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 & \vdots & & & & & \\
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r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow & r_n
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r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a,b)? \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n
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$$r_{n-3} = r_{n-2} q_{n-1} + r_{n-1} = (r_{n-1} q_n) q_{n-1} + r_{n-1} = r_{n-1}(q_n q_{n-1} + 1).$$

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Then

$$r_{n-4} = r_{n-3}q_{n-2} + r_{n-2}$$

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And so on...

Why does $r_{n-1} = \gcd(a, b)$?

Example: We saw

$$84 = 30 * 2 + 24$$
$$30 = 24 * 1 + 6$$
$$24 = 6 * 4 + 0.$$

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Example: We saw

$$\begin{aligned} 84 &= 30 * 2 + 24 \\ 30 &= 24 * 1 + 6 \\ 24 &= 6 * 4 + 0. \end{aligned} \qquad r_{n-1} = 6$$

So

$$30 = 24 * 1 + 6$$

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So

$$30 = 24 * 1 + 6 = (6 * 4) * 1 + 6 = 6(4 * 1 + 1) = 6 * 5$$

$$84 = 30 * 2 + 24 = (6 * 5) * 2 + (6 * 4)$$

Why does $r_{n-1} = \gcd(a, b)$?

Example: We saw

$$
\begin{aligned}
84 &= 30 * 2 + 24 \\
30 &= 24 * 1 + 6 \\
24 &= 6 * 4 + 0.
\end{aligned}
\qquad\qquad r_{n-1} = 6
$$

So

$$
\begin{aligned}
30 &= 24 * 1 + 6 = (6 * 4) * 1 + 6 = 6(4 * 1 + 1) = 6 * 5 \\
84 &= 30 * 2 + 24 = (6 * 5) * 2 + (6 * 4) = 6(5 * 2 + 4) = 6 * 24.
\end{aligned}
$$

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Example: We saw

$$84 = 30 * 2 + 24$$
$$30 = 24 * 1 + 6$$
$$24 = 6 * 4 + 0. \qquad r_{n-1} = 6$$

So

$$30 = 24 * 1 + 6 = (6 * 4) * 1 + 6 = 6(4 * 1 + 1) = 6 * 5$$
$$84 = 30 * 2 + 24 = (6 * 5) * 2 + (6 * 4) = 6(5 * 2 + 4) = 6 * 24.$$

So 6 is a common divisor of 84 and 30.

From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

$$\begin{aligned} 100 &= 36 * 2 + 28 \\ 36 &= 28 * 1 + 8 \\ 28 &= 8 * 3 + 4 \\ 8 &= 4 * 2 + 0. \end{aligned}$$

From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

$$\begin{aligned} 100 &= 36 * 2 + 28 \\ 36 &= 28 * 1 + 8 \\ 28 &= 8 * 3 + 4 \\ 8 &= 4 * 2 + 0. \end{aligned}$$

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So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$

From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

$$100 = 36 * 2 + 28$$
$$36 = 28 * 1 + 8$$
$$28 = 8 * 3 + 4$$
$$8 = 4 * 2 + 0. \qquad r_{n-1} = 4$$

So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$
$$36 = 28 * 1 + 8$$

From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

$$
\begin{aligned}
100 &= 36 * 2 + 28 \\
36 &= 28 * 1 + 8 \\
28 &= 8 * 3 + 4 \\
8 &= 4 * 2 + 0.
\end{aligned}
\qquad r_{n-1} = 4
$$

So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$

$$36 = 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2)$$

From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

$$\begin{aligned} 100 &= 36 * 2 + 28 \\ 36 &= 28 * 1 + 8 \\ 28 &= 8 * 3 + 4 \\ 8 &= 4 * 2 + 0. \end{aligned} \qquad r_{n-1} = 4$$

So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$
$$36 = 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2) = 4(7 * 1 + 2) = 4 * 9$$

From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

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So

$$\begin{aligned} 28 &= 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7 \\ 36 &= 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2) = 4(7 * 1 + 2) = 4 * 9 \\ 100 &= 36 * 2 + 28 \end{aligned}$$

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From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

$$100 = 36 * 2 + 28$$
$$36 = 28 * 1 + 8$$
$$28 = 8 * 3 + 4$$
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So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$
$$36 = 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2) = 4(7 * 1 + 2) = 4 * 9$$
$$100 = 36 * 2 + 28 = (4 * 9) * 2 + (4 * 7) = 4(9 * 2 + 7) = 4 * 25.$$

From our spreadsheet, we can calculate that for $a = 100$, $b = 36$:

$$\begin{aligned} 100 &= 36 * 2 + 28 \\ 36 &= 28 * 1 + 8 \\ 28 &= 8 * 3 + 4 \\ 8 &= 4 * 2 + 0. \end{aligned} \qquad r_{n-1} = 4$$

So

$$\begin{aligned} 28 &= 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7 \\ 36 &= 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2) = 4(7 * 1 + 2) = 4 * 9 \\ 100 &= 36 * 2 + 28 = (4 * 9) * 2 + (4 * 7) = 4(9 * 2 + 7) = 4 * 25. \end{aligned}$$

So 4 is a common divisor of 100 and 36.

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So 4 is a common divisor of 100 and 36.

You try: use the following computations, working backwards, to show that 2 is a common divisor of 100 and 26:

$$\begin{aligned} 100 &= 26 * 3 + 22 & 26 &= 22 * 1 + 4 \\ 22 &= 4 * 5 + 2 & 4 &= 2 * 2 + 0 \end{aligned}$$

Why does $r_{n-1} = \gcd(a, b)$?

Letting $r_{-1} = a$ and $r_0 = b$, and computing

$$
\begin{array}{rcccll}
r_{-1} & = & r_0 * q_1 & + & r_1 & \\
r_0 & = & r_1 * q_2 & + & r_2 & \\
r_1 & = & r_2 * q_3 & + & r_3 & \\
 & \vdots & & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a, b)? \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n
\end{array}
$$

we can reverse this process to show that r_{n-1} is, at the very least, a *common divisor* to $a = r_{-1}$ and $b = r_0$.

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 & \vdots & & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a, b)? \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n
\end{array}
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we can reverse this process to show that r_{n-1} is, at the very least, a *common divisor* to $a = r_{-1}$ and $b = r_0$.

Wait! How do we know we ever get 0??

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r_1 & = & r_2 * q_3 & + & r_3 \\
 & \vdots & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a, b)? \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n
\end{array}$$

we can reverse this process to show that r_{n-1} is, at the very least, a *common divisor* to $a = r_{-1}$ and $b = r_0$.

Wait! How do we know we ever get 0??

The division algorithm ensures that each remainder is strictly smaller than the last, and always non-negative:

$$b = r_0 > r_1 > r_2 > \cdots \geqslant 0.$$

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& \vdots & & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a,b)? \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n
\end{array}$$

we can reverse this process to show that r_{n-1} is, at the very least, a *common divisor* to $a = r_{-1}$ and $b = r_0$.

Wait! How do we know we ever get 0??

The division algorithm ensures that each remainder is strictly smaller than the last, and always non-negative:

$$b = r_0 > r_1 > r_2 > \cdots \geqslant 0.$$

So since the r_i's are all *integers*, this process ends at some point.

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We have that r_{n-1} is a common divisor to a an b. Now why is it the *greatest* common divisor?

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Suppose d is a common divisor of a and b, i.e. $d|a$ and $d|b$. This means

$$a = d\alpha \quad \text{and } b = d\beta \quad \text{for some } \alpha, \beta \in \mathbb{Z}.$$

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Back to our division calculation, and substitute these equations in:

$$\begin{array}{ccccc}
a & = & b * q_1 & + & r_1 \\
b & = & r_1 * q_2 & + & r_2 \\
r_1 & = & r_2 * q_3 & + & r_3 \\
 & \vdots & & & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} \\
r_{n-2} & = & r_{n-1} * q_n & + & 0
\end{array}$$

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Back to our division calculation, and substitute these equations in:

$$\begin{array}{ccccc} d\alpha & = & d\beta * q_1 & + & r_1 \\ b & = & r_1 * q_2 & + & r_2 \\ r_1 & = & r_2 * q_3 & + & r_3 \\ & \vdots & & & \\ r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} \\ r_{n-2} & = & r_{n-1} * q_n & + & 0 \end{array}$$

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Back to our division calculation, and substitute these equations in:

$$\begin{array}{ccccc}
d\alpha & = & d\beta * q_1 & + & r_1 \\
b & = & r_1 * q_2 & + & r_2 \\
r_1 & = & r_2 * q_3 & + & r_3 \\
 & \vdots & & & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} \\
r_{n-2} & = & r_{n-1} * q_n & + & 0
\end{array}$$

so $r_1 = d(\alpha - \beta q_1)$

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Theorem (Euclidean algorithm)

To compute the greatest common divisor of two positive integers a and b, let $r_{-1} = a$ and $r_0 = b$, and compute successive quotients and remainders

$$r_{i-2} = r_{i-1} q_i + r_i$$

for $i = 1, 2, 3, \ldots,$ until some remainder r_n is 0. The last nonzero remainder r_{n-1} is then the greatest common divisor of a and b.

This takes at most b steps (actually less), and is *much* more computationally efficient than the other methods.