From last time:

A Pythagorean triple is a triplet of positive integers $a, b, c \in \mathbb{Z}_{>0}$ satisfying $a^2 + b^2 = c^2$.

Ex: $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, and $8^2 + 15^2 = 17^2$.

Last time, we used factorization and divisors to help us prove the following.

1. If (a, b, c) is a Pythagorean triple, then so is (na, nb, nc) for any $n \in \mathbb{Z}_{\geqslant 0}$.
2. All primitive Pythagorean triples (those with no common divisors) are characterized by

$$a = st, \quad b = \frac{s^2 - t^2}{2}, \quad \text{and} \quad c = \frac{s^2 + t^2}{2},$$

for odd integers $s > t \geqslant 1$ with no common factors.

Today: Another approach, using geometry.

Pythagorean triples and the unit circle

For any $c \neq 0$, we have

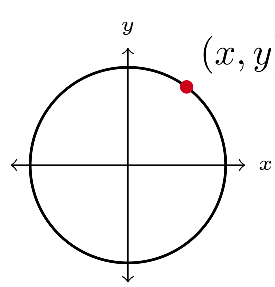
$$(a, b, c) \text{ is a solution to } \quad a^2 + b^2 = c^2$$

if and only if

$$(a, b, c) \text{ is a solution to } \quad \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

Let $x = a/c$ and $y = b/c$. Then solutions look like

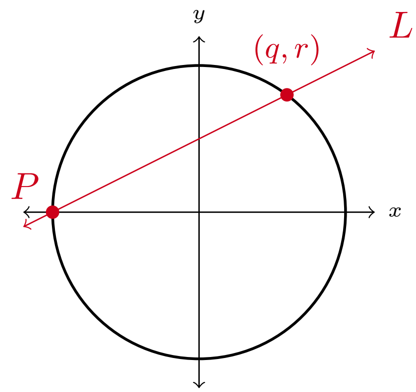
$x^2 + y^2 = 1$:



Integer solutions (a, b, c) occur whenever x and y are *rational*.

Pythagorean triples and the unit circle

$$x^2 + y^2 = 1$$



Integer solutions to $a^2 + b^2 = c^2$ occur whenever x and y are *rational*. (Let c be any common multiple of the denominators of x and y.)

Four obvious rational points: (1,0), (0,1), (-1,0), and (0, -1).

Take, for example, the point $P = (-1, 0)$.

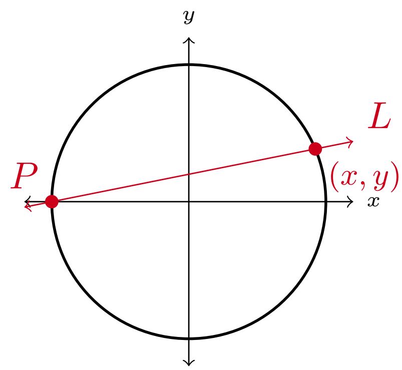
Now let (q, r) be any other rational point ($q, r \in \mathbb{Q}$) on the circle.

Consider the line L connecting those two points. Rational slope!

Pythagorean triples and the unit circle

If we take a line through $P = (-1, 0)$ and another rational point (q, r) on the unit circle, that line will have rational slope.

$$x^2 + y^2 = 1$$



Conversely, take any line with rational slope m that intersects P,

$$L : y = m(x + 1), \quad m \in \mathbb{Q}$$

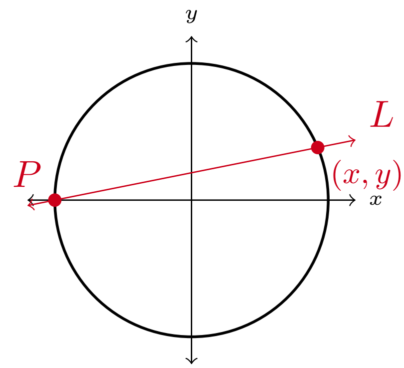
(using point-slope formula).
Let (x, y) be the other point where the line intersects the circle.
Solve.

Pythagorean triples and the unit circle

$$x^2 + y^2 = 1$$

$$L : y = m(x+1), \quad m \in \mathbb{Q}$$

Two points of intersection:
$(-1, 0)$ and $\left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$



Both rational!!

Theorem

Every point on the circle $x^2 + y^2 = 1$ whose coordinates are rational numbers can be obtained from the formula

$$(x, y) = \left(\frac{1-m^2}{1+m^2}, \frac{2m}{1+m^2}\right)$$

by substituting in rational numbers for m or taking the limit $m \to \infty$.

Theorem

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Relating back to last time: rational points (x, y) on the unit circle correspond to primitive Pythagorean triples (a, b, c) as follows:

Process:

- Put x and y into lowest terms.
- Let c be the smallest common multiple of their denominators.
- Let $a = xc$ and $b = yc$

Example:

- $(x, y) = (3/5, 4/5)$
- $c = 5$
- $a = 3$ and $b = 4$

Last time: $(a, b, c) = (st, \frac{1}{2}(s^2 - t^2), \frac{1}{2}(s^2 + t^2))$

Substitute $m = u/v$. Then let $u = \frac{1}{2}(s + t)$, $v = \frac{1}{2}(s - t)$.