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Today: Another approach, using geometry.

Pythagorean triples and the unit circle

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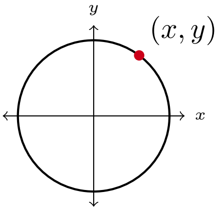

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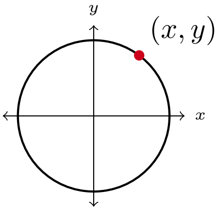

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Pythagorean triples and the unit circle

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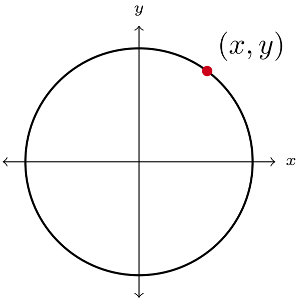

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Pythagorean triples and the unit circle

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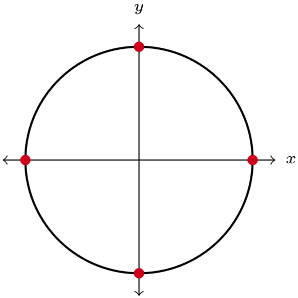

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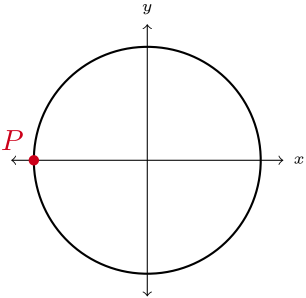

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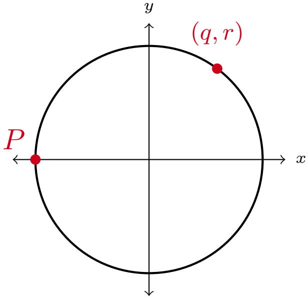

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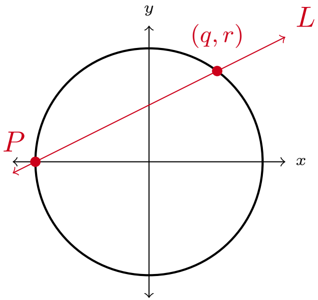

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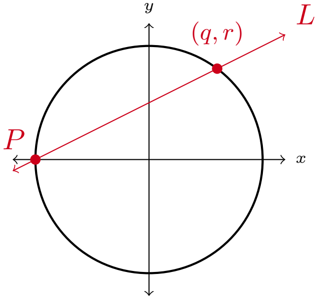

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Consider the line L connecting those two points. Rational slope!

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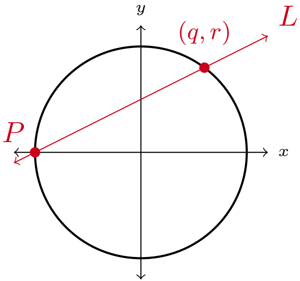

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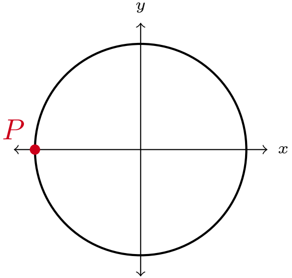

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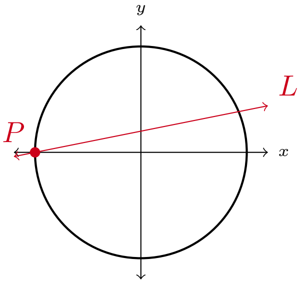

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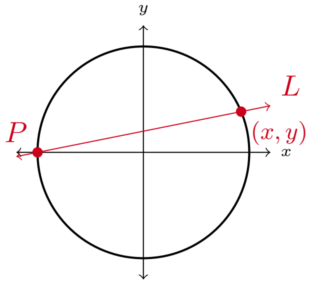

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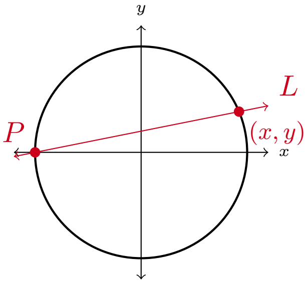

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Solve.

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Two points of intersection:
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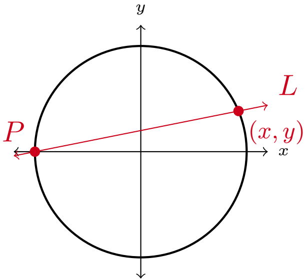

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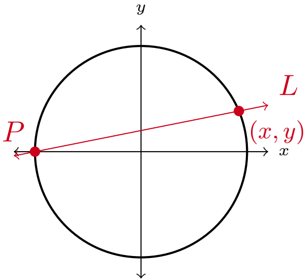

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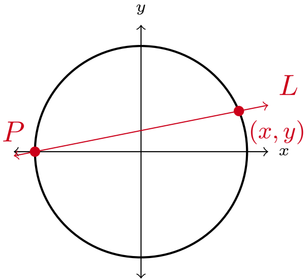

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Every point on the circle $x^2 + y^2 = 1$ whose coordinates are rational numbers can be obtained from the formula

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