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Ex: Since $1 + i$ divides 2, and it is not of the form $2u$ or u for any unit u, 2 is not prime.

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$$N : \mathbb{Z}[i] \to \mathbb{Z}_{\geqslant 0} \quad \text{by} \quad a + bi \mapsto a^2 + b^2.$$

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The units in $\mathbb{Z}[i]$ are $\{\pm 1, \pm i\}$.

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Back to primes: Is 2 prime in $\mathbb{Z}[i]$?

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So since $1 + i$ isn't a unit, nor is it a unit multiple of 2, we have 2 is not prime in $\mathbb{Z}[i]$!!

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Let p be an odd prime. Then there are integers a, b satisfying

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So 3 is a prime in $\mathbb{Z}[i]$.

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1. If $n \in \mathbb{Z}$ is composite in $\mathbb{Z}$, then it is composite in $\mathbb{Z}[i]$.
2. If $p \in \mathbb{Z}$ is prime in $\mathbb{Z}$, then either
   (a) $p = 2$, which is not prime in $\mathbb{Z}[i]$; (we checked)
   (b) $p \equiv_4 -1$, in which case p is prime in $\mathbb{Z}[i]$; (prove using norms)
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Are there any more?

Theorem (Gaussian Prime Theorem)

The Gaussian primes can be described as follows:

(i) *(ramified) 1 + i is a Gaussian prime.*

(ii) *(inert) Let p be a prime in $\mathbb{Z}$ with $p \equiv -1 \pmod 4$. Then p is a Gaussian prime.*

(iii) *(split) Let p be a prime in $\mathbb{Z}$ with $p \equiv 1 \pmod 4$. Then $p = a^2 + b^2$ for $a, b \in \mathbb{Z}_{>0}$, and $a + bi$ is a Gaussian prime.*

Moreover, every Gaussian prime is equal to a unit times a Gaussian prime of the form (i), (ii), or (iii).

We can also use $N(\alpha)$ to find divisors of α.

Lemma (Gaussian Divisibility Lemma)

Let $\alpha \in \mathbb{Z}[i]$.

(a) *If 2 divides $N(\alpha)$ in $\mathbb{Z}$, then $1+i$ divides α in $\mathbb{Z}[i]$.*

(b) *Let p be (an inert) prime, and suppose that p divides $N(\alpha)$ in $\mathbb{Z}$. Then p divides α in $\mathbb{Z}[i]$.*

(c) *Let $\pi = u + vi$ be a split, and let $\overline{\pi} = u - vi$. If $N(\pi)$ divides $N(\alpha)$ in $\mathbb{Z}$, then at least one of π or $\overline{\pi}$ divides α in $\mathbb{Z}[i]$.*

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To be clear:
a divides b in $\mathbb{Z}$ if there is a $k \in \mathbb{Z}$ such that $ak = b$.
α divides β in $\mathbb{Z}[i]$ if there is a $\gamma \in \mathbb{Z}[i]$ such that $\alpha\gamma = \beta$.