Using logarithms to do computations

Fix $p = 37$. Then 2 is a primitive root.

The discrete logarithm values are given by the following.

b	1	2	3	4	5	6	7	8	9
$\text{dlog}_2(b)$	36	1	26	2	23	27	32	3	16

b	10	11	12	13	14	15	16	17	18
$\text{dlog}_2(b)$	24	30	28	11	33	13	4	7	17

b	19	20	21	22	23	24	25	26	27
$\text{dlog}_2(b)$	35	25	22	31	15	29	10	12	6

b	28	29	30	31	32	33	34	35	36
$\text{dlog}_2(b)$	34	21	14	9	5	20	8	19	18

Example: Use the logarithm table to compute the following $\pmod{37}$:

(1) $25 \cdot 16$ (2) 28^{32} (3) 9^{-1}

(4) x satisfying $20x \equiv 3$ (5) $3x^{30} \equiv 4$

Chapter 35: Number Theory and Imaginary Numbers

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Try: Compute $(2 + 3i)^3$, $(2 + 3i)(-1 + 4i)$, $\dfrac{2 + 3i}{-1 + 4i}$, and $\dfrac{5 - i}{1 + 2i}$.

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$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

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For $m, n \in \mathbb{Z}[i]$, we say m divides n if there is some $k \in \mathbb{Z}[i]$ such that $mk = n$.

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Every integer $n \geqslant 2$ has a unique factorization into primes.

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We'll show there are no more solutions momentarily.

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Primes in $\mathbb{Z}[i]$? For any $\alpha \in \mathbb{Z}[i]$, we have

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so $\pm 1, \pm i$, $\pm \alpha$, and $\pm i\alpha$ all "divide" α.

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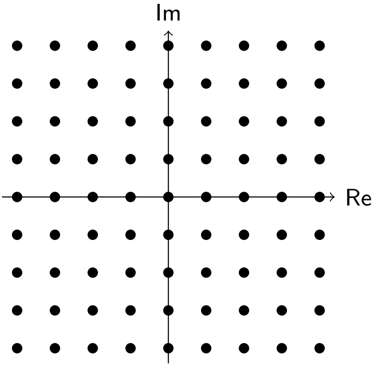
How do we compute primes?

Draw $\mathbb{Z}[i]$ in the complex plane as a lattice of points:



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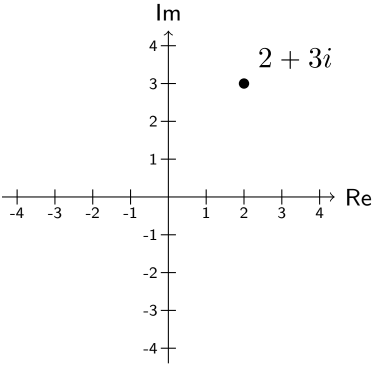

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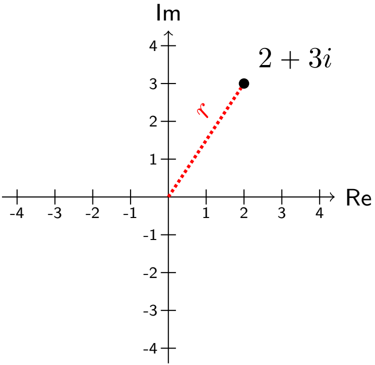

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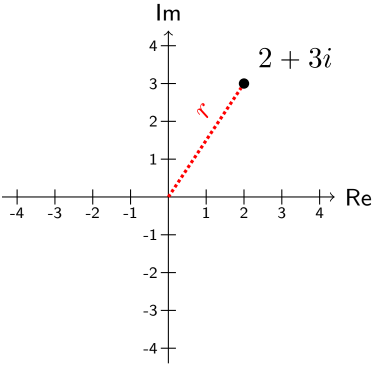

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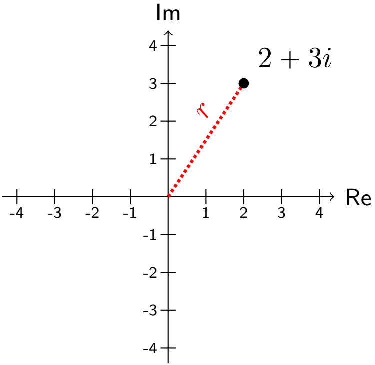

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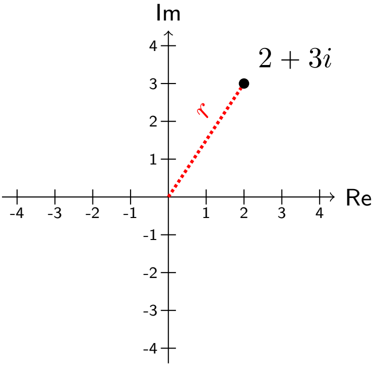

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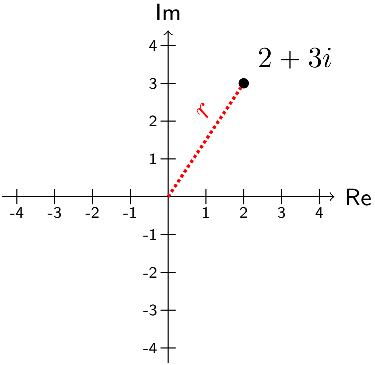

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We call N a norm of $\mathbb{Z}[i]$.

Norm

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So $N(\alpha) = N(\beta) = 1$. What are integer solutions to $a^2 + b^2 = 1$?

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Theorem

The units in $\mathbb{Z}[i]$ are $\{\pm 1, \pm i\}$.

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Back to primes: Is 2 prime in $\mathbb{Z}[i]$?

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Taking N of both sides, we get

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Are there non-trivial solutions to $a^2 + b^2 = 2$?

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$a^2 + b^2 = 1$: In this case, $a + bi$ is a unit.
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$$\frac{2}{1 + i} = \frac{2(1 - i)}{2} = 1 - i.$$
So since $1 + i$ isn't a unit, nor is it a unit multiple of 2, we have 2 is not prime in $\mathbb{Z}[i]$!!

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So 3 is a prime in $\mathbb{Z}[i]$.

We say $\beta \in \mathbb{Z}[i]$ is prime if the only divisors of β are of the form u or $u\beta$, where u is a unit (one of $\{\pm 1, \pm i\}$).

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Are there any more?

Theorem (Gaussian Prime Theorem)

The Gaussian primes can be described as follows:

(i) *(ramified) 1 + i is a Gaussian prime.*

(ii) *(inert) Let* p *be a prime in* $\mathbb{Z}$ *with* $p \equiv -1 \pmod 4$*. Then* p *is a Gaussian prime.*

(iii) *(split) Let* p *be a prime in* $\mathbb{Z}$ *with* $p \equiv 1 \pmod 4$*. Then* $p = a^2 + b^2$ *for* $a, b \in \mathbb{Z}_{>0}$*, and* $a + bi$ *is a Gaussian prime.*

Moreover, every Gaussian prime is equal to a unit times a Gaussian prime of the form (i), (ii), or (iii).