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2	2	4	1	2	4	1
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You try: Compute $\psi_p(k)$ for $1 \leqslant k \leqslant p - 1$ for $p = 3, 5$, and 7.