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$$\text{QR} \times \text{QR} = \text{QR} \quad \text{QR} \times \text{NR} = \text{NR} \quad \text{NR} \times \text{NR} = \text{QR}.$$

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The Legendre symbol of a modulo p is

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a QR,} \\ -1 & \text{if } a \text{ is a NR,} \\ 0 & \text{if } a \text{ is a multiple of } p. \end{cases}$$

So

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right).$$

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Let p be an odd prime. Then

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Strategy: use $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$, and compute $\left(\frac{a}{p}\right)$ for small values, like primes.

$\leftarrow q \rightarrow$

$\uparrow p \downarrow$

$\left(\frac{q}{p}\right)$	3	5	7	11	13	17	19	23
3	0	−1	1	−1	1	−1	1	−1
5	−1	0	−1	1	−1	−1	1	−1
7	−1	−1	0	1	−1	−1	−1	1
11	1	1	−1	0	−1	−1	−1	1
13	1	−1	−1	−1	0	1	−1	1
17	−1	−1	−1	−1	1	0	1	−1
19	−1	1	1	1	−1	1	0	1
23	1	−1	−1	−1	1	−1	−1	0

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$\left(\frac{q}{p}\right)$	3	5	7	11	13	17	19	23
3	0	$\boxed{-1}$	①	$\boxed{-1}$	①	$\boxed{-1}$	①	$\boxed{-1}$
5	$\boxed{-1}$	0	$\boxed{-1}$	①	$\boxed{-1}$	$\boxed{-1}$	①	$\boxed{-1}$
7	$\boxed{-1}$	$\boxed{-1}$	0	①	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	①
11	①	①	$\boxed{-1}$	0	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	①
13	①	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	0	①	$\boxed{-1}$	①
17	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	①	0	①	$\boxed{-1}$
19	$\boxed{-1}$	①	①	①	$\boxed{-1}$	①	0	①
23	①	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	①	$\boxed{-1}$	$\boxed{-1}$	0

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$\left(\frac{q}{p}\right)$	3	5	7	11	13	17	19	23
3	0	$\boxed{-1}$	$\textcircled{1}$	$\boxed{-1}$	$\textcircled{1}$	$\boxed{-1}$	$\textcircled{1}$	$\boxed{-1}$
5	$\boxed{-1}$	0	$\boxed{-1}$	$\textcircled{1}$	$\boxed{-1}$	$\boxed{-1}$	$\textcircled{1}$	$\boxed{-1}$
7	$\boxed{-1}$	$\boxed{-1}$	0	$\textcircled{1}$	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	$\textcircled{1}$
11	$\textcircled{1}$	$\textcircled{1}$	$\boxed{-1}$	0	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	$\textcircled{1}$
13	$\textcircled{1}$	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	0	$\textcircled{1}$	$\boxed{-1}$	$\textcircled{1}$
17	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	$\textcircled{1}$	0	$\textcircled{1}$	$\boxed{-1}$
19	$\boxed{-1}$	$\textcircled{1}$	$\textcircled{1}$	$\textcircled{1}$	$\boxed{-1}$	$\textcircled{1}$	0	$\textcircled{1}$
23	$\textcircled{1}$	$\boxed{-1}$	$\boxed{-1}$	$\boxed{-1}$	$\textcircled{1}$	$\boxed{-1}$	$\boxed{-1}$	0

blue: $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right)$ **red:** $\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right)$

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$\left(\frac{q}{p}\right)$	3	5	7	11	13	17	19	23
3	0	−1 (blue)	1 (red)	−1 (red)	1 (blue)	−1 (blue)	1 (red)	−1 (red)
5	−1 (blue)	0	−1 (blue)	1 (blue)	−1 (blue)	−1 (blue)	1 (blue)	−1 (blue)
7	−1 (red)	−1 (blue)	0	1 (red)	−1 (blue)	−1 (blue)	−1 (red)	1 (red)
11	1 (red)	1 (blue)	−1 (red)	0	−1 (blue)	−1 (blue)	−1 (red)	1 (red)
13	1 (blue)	−1 (blue)	−1 (blue)	−1 (blue)	0	1 (blue)	−1 (blue)	1 (blue)
17	−1 (blue)	−1 (blue)	−1 (blue)	−1 (blue)	1 (blue)	0	1 (blue)	−1 (blue)
19	−1 (red)	1 (blue)	1 (red)	1 (red)	−1 (blue)	1 (blue)	0	1 (red)
23	1 (red)	−1 (blue)	−1 (red)	−1 (red)	1 (blue)	−1 (blue)	−1 (red)	0

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Cols/rows that are all blue:

$$p, q = 5, 13, 17$$

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Cols/rows that are all blue:

$$p, q = 5, 13, 17, 29, 37, 41, \cdots \equiv_4 1$$

Theorem (Quadratic reciprocity, primes)

Let p and q be odd primes. Then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv_4 1, \\ -1 & \text{if } p \equiv_4 -1, \end{cases} \qquad \left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv_8 \pm 1, \\ -1 & \text{if } p \equiv_8 \pm 3, \end{cases}$$

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(Presented without proof)

Example: Is 350 a perfect square modulo 11?

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You try: Compute

$$\left(\frac{20}{31}\right), \quad \left(\frac{14}{137}\right), \quad \text{and} \quad \left(\frac{55}{179}\right).$$

(31, 137, and 179 are prime)

Thm. For a composite number $b = p_1 p_2 \cdots p_\ell$, we have

$$\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right)\cdots\left(\frac{a}{p_\ell}\right).$$

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Theorem (Quadratic reciprocity, composites)

Let a and b be odd positive integers. Then

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