$b^2 \pmod{n}$:

$\leftarrow n \rightarrow$

b	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1
2	4	4	4	4	4	4	4	4	4
3	4	3	2	1	0	9	9	9	9
4	1	4	2	0	7	6	5	4	3
5		1	4	1	7	5	3	1	12
6			1	4	0	6	3	0	10
7				1	4	9	5	1	10
8					1	4	9	4	12
9						1	4	9	3
10							1	4	9
11								1	4
12									1

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2	4	4	4	4	4	4	4	4	4
3	4	3	2	1	0	9	9	9	9
4	1	4	2	0	7	6	5	4	3
5		1	4	1	7	5	3	1	12
6			1	4	0	6	3	0	10
7				1	4	9	5	1	10
8					1	4	9	4	12
9						1	4	9	3
10							1	4	9
11								1	4
12									1

modulo 13:

b	b'	b^2
1	1	1
2	2	4
3	3	9
4	4	3
5	5	12
6	6	10
7	-6	10
8	-5	12
9	-4	3
10	-3	9
11	-2	4
12	-1	1

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$\leftarrow \; n \; \rightarrow$

b	5	6	7	8	9	10	11	12	13
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5		1	4	1	7	5	3	1	12
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10							1	4	9
11								1	4
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modulo 13:

b	b'	b^2
1	1	1
2	2	4
3	3	9
4	4	3
5	5	12
6	6	10
7	− 6	10
8	− 5	12
9	− 4	3
10	− 3	9
11	− 2	4
12	− 1	1

Most values will appear at least twice: $b^2 = (-b)^2 \equiv_n (n-b)^2$.

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b	5	6	7	8	9	10	11	12	13
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2	4	4	4	4	4	4	4	4	4
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Theorem. Let p be an odd prime. Then there are exactly $(p-1)/2$ quadratic residues modulo p and exactly $(p-1)/2$ nonresidues modulo p. (Namely, there are as many residues as possible, which is half.)

Arithmetic with quadratic residues

QR $\times$ QR: Suppose a and a' are QRs modulo p.

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But we have some b, b' such that $b^2 \equiv_p a$ and $(b')^2 \equiv_p a'$.

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Since $p \nmid a$ and $p \nmid a'$, we have $p \nmid aa'$.
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Moreover, we have some b such that $b^2 \equiv_p a$.

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which is a contradiction.

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Consider

$$a,\ 2a,\ \ldots,\ (p-1)a \qquad \pmod{p}.$$

Since $p \nmid a$, we have $\gcd(a,p)=1$, so as we showed in proving Fermat's Little Theorem, this list is just a rearrangement of

$$1,\ 2,\ \ldots,\ (p-1) \qquad \pmod{p}.$$

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In particular, this list has the $(p-1)/2$ QRs and the $(p-1)/2$ NRs!

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$$\{1,2,\ldots,p-1\} \to \{1,2,\ldots,p-1\} \quad \text{defined by} \quad x \mapsto ax \pmod{p}$$

sends the $(p-1)/2$ QRs to (distinct) NRs.

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In particular, this list has the $(p-1)/2$ QRs and the $(p-1)/2$ NRs! But we showed that QR × NR = NR. So

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sends the $(p-1)/2$ QRs to (distinct) NRs. Therefore, it *must* send the $(p-1)/2$ NRs all to QRs.

Arithmetic with quadratic residues

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Since $p \nmid a$, we have $\gcd(a,p)=1$, so as we showed in proving Fermat's Little Theorem, this list is just a rearrangement of

$$1,\ 2,\ \ldots,\ (p-1) \qquad \pmod{p}.$$

In particular, this list has the $(p-1)/2$ QRs and the $(p-1)/2$ NRs! But we showed that QR × NR = NR. So

$$\{1,2,\ldots,p-1\} \to \{1,2,\ldots,p-1\} \quad \text{defined by} \quad x \mapsto ax \pmod{p}$$

sends the $(p-1)/2$ QRs to (distinct) NRs. Therefore, it *must* send the $(p-1)/2$ NRs all to QRs.

In other words, NR × NR = QR.

Arithmetic with quadratic residues: Legendre symbol

We have

$$\text{QR} \times \text{QR} = \text{QR} \quad \text{NR} \times \text{QR} = \text{QR} \quad \text{NR} \times \text{NR} = \text{QR}.$$

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The Legendre symbol of a modulo p is

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a QR,} \\ -1 & \text{if } a \text{ is a NR,} \\ 0 & \text{if } a \text{ is a multiple of } p. \end{cases}$$

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Theorem (Quadratic Residue Multiplication Rule)

Let p be a prime. Then

$$\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right).$$

Spotting small QRs

If $p = 2$, then the possible residues are 0 and 1.

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Now consider the equation $x^N - 1 \equiv_p 0$ for $N = (p-1)/2$. Since p is prime, there are at *most* $(p-1)/2$ solutions. Also, every one of the $(p-1)/2$ quadratic residues are solutions.

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□

Theorem

If p is an odd prime then

$$a^{(p-1)/2} \equiv_p \left(\frac{a}{p}\right).$$

Let

$$A = a^{(p-1)/2} \quad \text{(reduced modulo } p\text{)}.$$

Example:

Recall, modulo 13, the QRs are 1, 3, 4, 9, 10, and 12.

a	1	2	3	4	5	6	7	8	9	10	11	12
$\left(\frac{a}{13}\right)$	1	-1	1	1	-1	-1	-1	-1	1	1	-1	1
A	1	12	1	1	12	12	12	12	1	1	12	1

Theorem

If p is an odd prime then

$$a^{(p-1)/2} \equiv_p \left(\frac{a}{p}\right).$$

Corollary (Quadratic reciprocity)

Let p be an odd prime. Then

$$\left(\frac{-1}{p}\right) = \begin{cases} 1 & \text{if } p \equiv_4 1, \\ -1 & \text{if } p \equiv_4 -1. \end{cases}$$

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Proof.

Compute $(-1)^{(p-1)/2} \pmod{p}$. □

$b^2 \pmod p$:

$\leftarrow \; p \; \rightarrow$

b	3	5	7	11	13
1	1	1	1	1	1
2	1	$\boxed{4}$	4	4	4
3		$\boxed{4}$	2	9	9
4		1	2	5	3
5			4	3	$\boxed{12}$
6			1	3	10
7				5	10
8				9	$\boxed{12}$
9				4	3
10				1	9
11					4
12					1

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7				5	10
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12					1

When is 2 a quadratic residue? (Read Chapter 21)

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When is 2 a quadratic residue? (Read Chapter 21)

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So since $\gcd(P!, p) = 1$, we have $(-1)^N \equiv_p 2^P$.

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Compute N... □