Density of primes

Question: What is the density of primes in $\mathbb{Z}_{>0}$?

More concrete example: What is the density of the even integers amongst all positive integers?

Intuitively, about half. How to calculate?
Start with finite sets:

S	$\{x \in S \mid x$ even $\}$	Density
$\{1\}$	$\varnothing$	$0/1 = 0$
$\{1,2\}$	$\{2\}$	$1/2 = .5$
$\{1,2,3\}$	$\{2\}$	$1/3 \approx .3$
$\{1,2,3,4\}$	$\{2,4\}$	$2/4 = .5$
$\{1,\ldots,5\}$	$\{2,4\}$	$2/5 = .4$
$\{1,\ldots,6\}$	$\{2,4,6\}$	$3/6 = .5$
$\{1,\ldots,7\}$	$\{2,4,6\}$	$3/7 \approx .43$
$\{1,\ldots,n\}$	$\{2,4,\ldots,2\lfloor n/2\rfloor\}$	$\lfloor n/2\rfloor/n$

Then the density is the limit as $n \to \infty$:

$$\lim_{n\to\infty} \lfloor n/2\rfloor/n = 1/2.$$

Density of primes

Question: What is the density of primes in $\mathbb{Z}_{>0}$?
Let $\pi(n) = \#\{$ primes $p \leqslant n\}$.

n	$\pi(n)$	Density	$n/\ln n$
10	4	$4/10 = .4$	≈ 4.3
25	9	$9/25 = .36$	≈ 7.8
50	15	$15/50 = .3$	≈ 12.8
100	25	$25/100 = .25$	≈ 21.7
500	95	.19	≈ 80.5
1000	168	.168	≈ 144.8
5000	669	.134	≈ 587.0

As $n \to \infty$, we have $\pi(n)/n \to 0$.

Theorem (Prime number theorem)

As $n \to \infty$, we have $\pi(n) \to n/\ln(n)$. In other words,

$$\lim_{n\to\infty} \frac{\pi(n)}{n/\ln(n)}.$$

(Proof in Analytic Number Theory, using complex analysis.)

Other theorems/open questions in analytic number theory

Goldbach's Conjecture: Every even $n \geqslant 4$ is a sum of two primes.

Proven: Every (sufficiently large) odd n is a sum of three primes.

Twin Primes Conjecture: There are infinitely many prime numbers p such that $p+2$ is also prime.

Proven: There are infinitely many prime numbers p such that $p+2$ is either prime or the product of 2 primes.

Conjecture: There are infinitely many prime numbers of the form N^2+1, with $N \in \mathbb{Z}$.

Proven: There are infinitely many (even) N such that N^2+1 is either prime or the product of two primes.

Let

$$T(n) = \#\{ \text{ primes } p \leqslant n \text{ such that } p+2 \text{ is also prime } \},$$
$$S(x) = \#\{\text{primes } p \leqslant n \text{ such that } p = N^2+1 \text{ for some } N \in \mathbb{Z}\}$$

Conjecture: Both

$$\lim_{n\to\infty} \frac{T(n)}{n/(\ln(n))^2} \quad \text{and} \quad \lim_{n\to\infty} \frac{S(n)}{\sqrt{n}/\ln(n)}$$

exist and are positive.

Other theorems/open questions in analytic number theory

Analytic number theory: studying number theory using advanced calculus (analysis).