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S	$\{x \in S \mid x \text{ even }\}$	Density
$\{1\}$	$\varnothing$	$0/1 = 0$
$\{1,2\}$	$\{2\}$	$1/2 = .5$
$\{1,2,3\}$	$\{2\}$	$1/3 \approx .3$
$\{1,2,3,4\}$	$\{2,4\}$	$2/4 = .5$
$\{1,\ldots,5\}$	$\{2,4\}$	$2/5 = .4$
$\{1,\ldots,6\}$	$\{2,4,6\}$	$3/6 = .5$
$\{1,\ldots,7\}$	$\{2,4,6\}$	$3/7 \approx .43$

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$\{1,\ldots,n\}$	$\{2,4,\ldots,2\lfloor n/2\rfloor\}$	$\lfloor n/2\rfloor/n$

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Then the density is the limit as $n \to \infty$:

$$\lim_{n\to\infty} \lfloor n/2\rfloor/n = 1/2.$$

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n	$\pi(n)$	Density
10	4	$4/10 = .4$
25	9	$9/25 = .36$
50	15	$15/50 = .3$
100	25	$25/100 = .25$
500	95	$.19$
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(Proof in Analytic Number Theory, using complex analysis.)

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Conjecture: Both

$$\lim_{n\to\infty} \frac{T(n)}{n/(\ln(n)^2} \quad \text{and} \quad \lim_{n\to\infty} \frac{S(n)}{\sqrt{n}/\ln(n)}$$

exist and are positive.