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Theorem
There are infinitely many primes.

Arithmetic progressions

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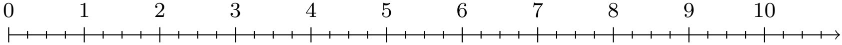

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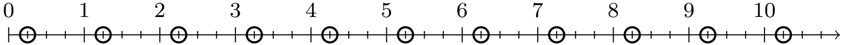

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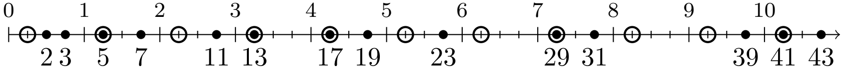

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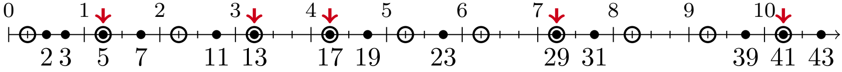

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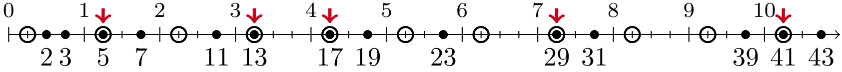

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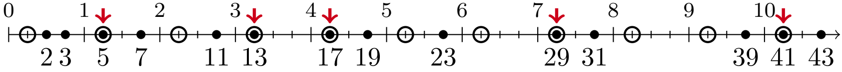

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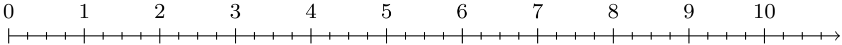

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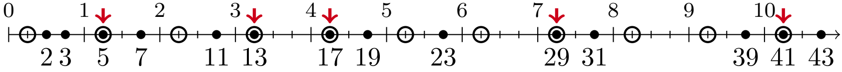

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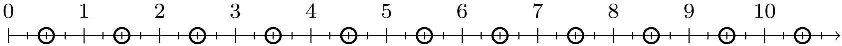

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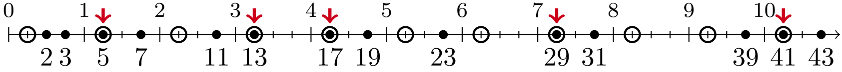

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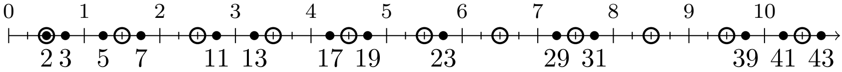

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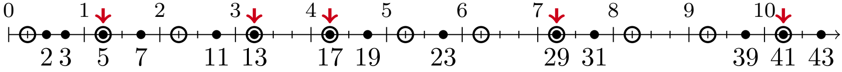

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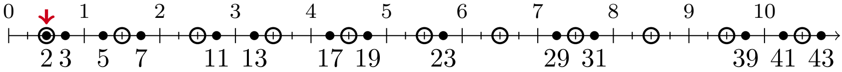

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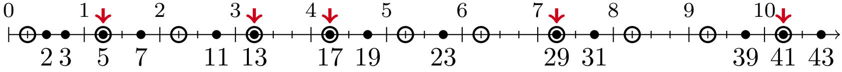

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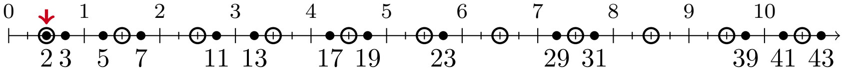

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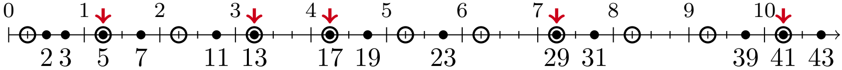

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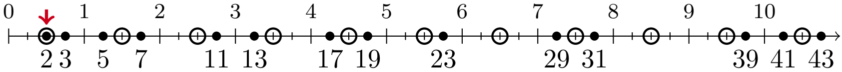

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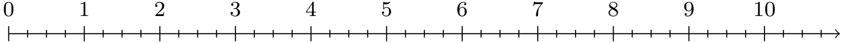

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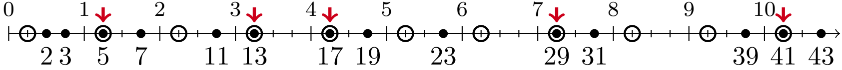

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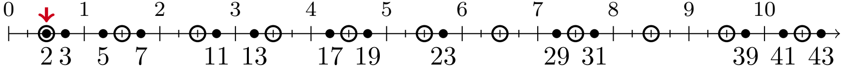

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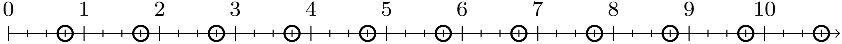

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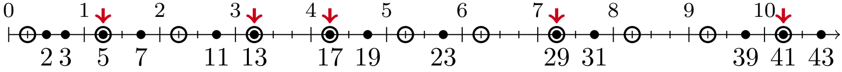

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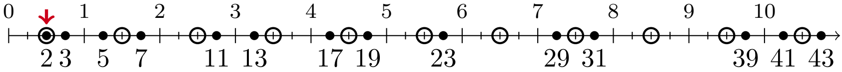

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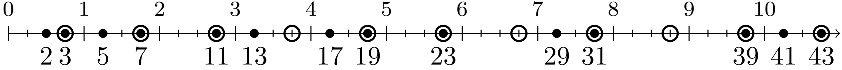

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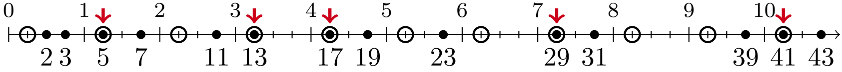

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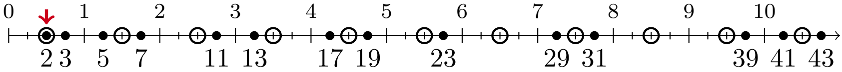

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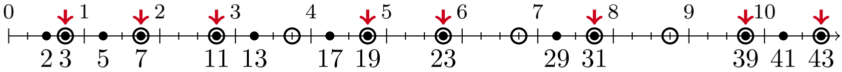

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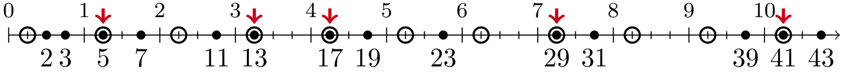

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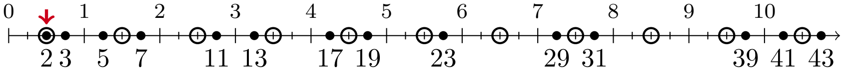

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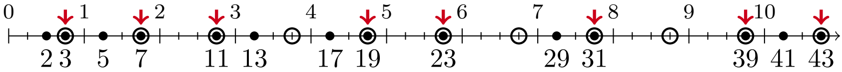

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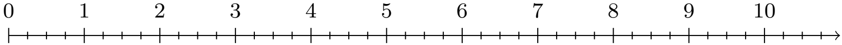

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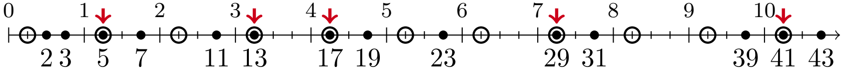

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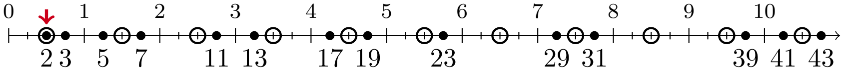

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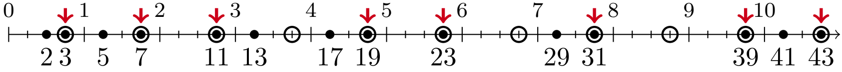

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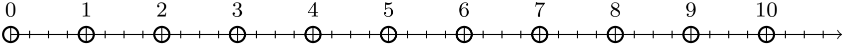

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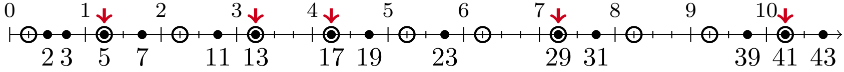

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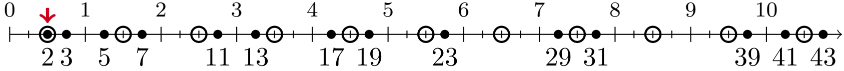

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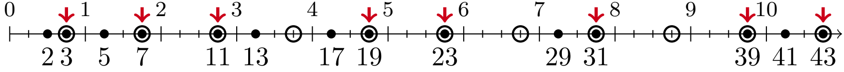

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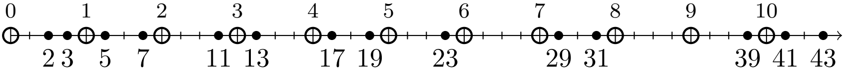

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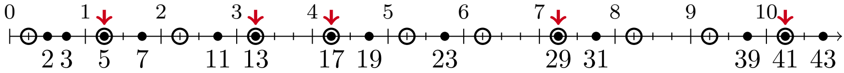

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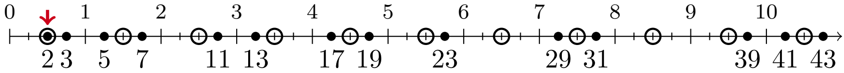

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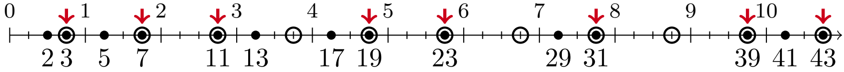

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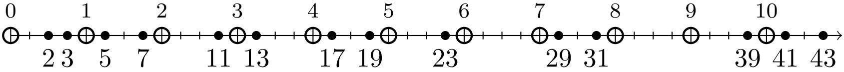

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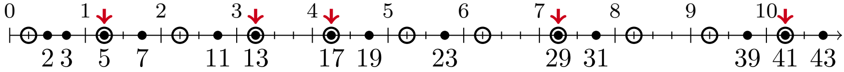

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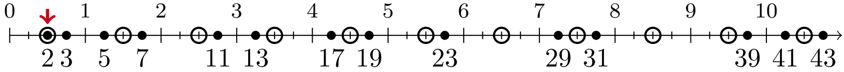

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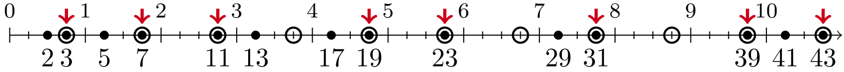

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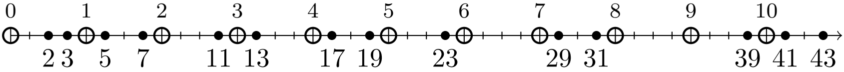

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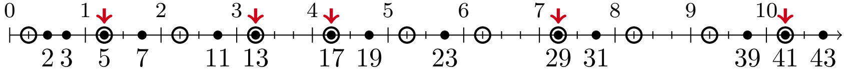

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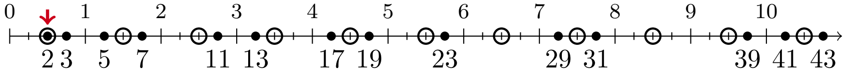

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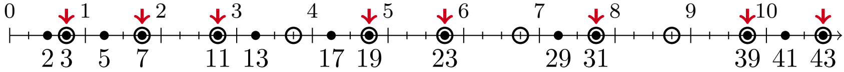

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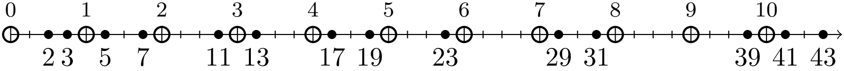

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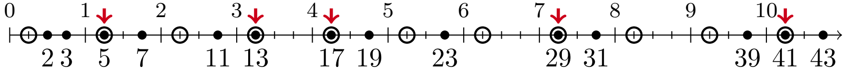

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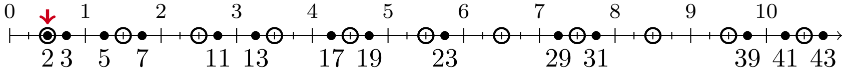

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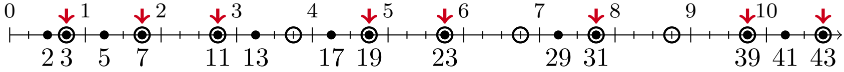

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Fact: There are no primes congruent to $0 \pmod 4$, and there is exactly 1 prime congruent to $2 \pmod 4$.

Hypothesis: There are infinitely many primes that are congruent to $1 \pmod 4$, and there are infinitely many primes that are congruent to $3 \pmod 4$.

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Recall, an arithmetic progression (or arithmetic sequence) is a sequence of numbers that differ by a constant value n

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Why doesn't this proof work for showing that there are infinitely many primes that are congruent to $3 \pmod 4$?