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Goal: Show $|\Phi(mn)| = |\Phi(m) \times \Phi(n)|$ by giving a bijection

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Let m *and* n *be integers satisfying* $\gcd(m,n) = 1$*, and let* b *and* c *be any integers. Then the simultaneous congruences*

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$$y \equiv_5 4^3(4y) \equiv_5 4^3 \cdot 2 = 128 \equiv_5 3.$$

So $x = 1 + 4 \cdot 3 = \boxed{13}$.

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Let m and n be integers satisfying $\gcd(m,n)=1$, and let b and c be any integers. Then the simultaneous congruences

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Back to the ϕ function

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If $\gcd(m, n) = 1$*, then* $\phi(mn) = \phi(m)\phi(n)$*.*

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Theorem

Let $\phi(n)$ be the number of integers relatively prime to n (up to equivalence). Then $\phi(n)$ can be calculated by

1. *if p is prime, then $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1)$; and*
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Consider what it means that $x^{12002} \equiv 9 \pmod{7000}$:
This is equivalent to

$$x^{12002} - 9 = 7000k \quad \text{for some } k \in \mathbb{Z}, \quad \text{i.e. } 9 = x^{12002} - 7000k.$$

So since 9 is an integer combination of x and 7000, we must have $\gcd(x, 7000) | 9$. But $\gcd(7000, 9) = 1$, so the only possibility is $\gcd(x, 7000) = 1$.
Yes, x relatively prime to 7000! So we can use $x^{\phi(n)} \equiv 1 \pmod{n}$...

We just saw

$$\phi(7000) = (2^3 - 2^2)(5^3 - 5^2)(7 - 1) = \boxed{2400}.$$

So since

$$12002 = 5(2400) + 2, \quad \text{we have } x^{12002} = (x^{2400})^5 x^2.$$

So

$$9 \equiv_{7000} x^{12002} = (x^{2400})^5 x^2 \equiv_{7000} 1^5 x^2 = x^2.$$

At least 2 sol's: $x = 3$ and $x = -3 \equiv_{7000} 6997$. (There may be more.)