Solutions
Math 347
Homework 1
9/6/17

Exercise 1. When we take a composite number n and "factor it into primes", that means we write it as a product of prime numbers, usually in increasing order, using exponents to simplify:

$$n = p_1^{m_1} p_2^{m_2} \cdots p_\ell^{m_\ell}, \qquad p_1 < p_2 < \cdots < p_\ell \quad \text{primes.}$$

For example,

$$4840 = 2^3 \cdot 5 \cdot 11^2, \qquad 161 = 7 \cdot 23, \qquad 19683 = 3^9.$$

List the first 15 composite numbers, and factor them into primes.

Answer.

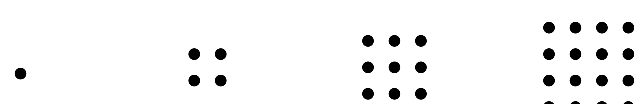
$$\begin{aligned}
4 &= 2^2 \\
6 &= 2 * 3 \\
8 &= 2^3 \\
9 &= 3^2 \\
10 &= 2 * 5 \\
12 &= 2^2 * 3 \\
14 &= 2 * 7 \\
15 &= 3 * 5 \\
16 &= 2^4 \\
18 &= 2 * 3^2 \\
20 &= 2^2 * 5 \\
21 &= 3 * 7 \\
22 &= 2 * 11 \\
24 &= 2^3 * 3 \\
25 &= 5^2
\end{aligned}$$

□

Exercise 2. Square and triangular numbers

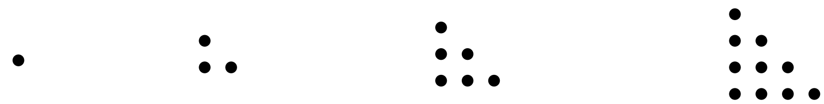
Square numbers are any number of the form n^2 for $n \in \mathbb{Z}_{>0}$:

$1\ (= 1^2) \qquad 4\ (= 2^2) \qquad 9\ (= 3^2) \qquad 16\ (= 4^2)$



Triangular numbers are any number of the form $1 + 2 + 3 + \cdots + n$ for $n \in \mathbb{Z}_{>0}$:

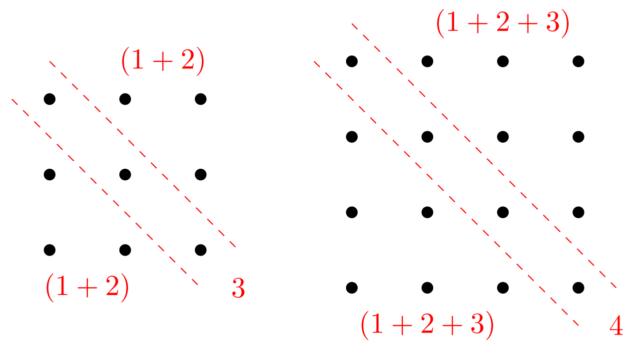
$1\ (= 1) \qquad 3\ (= 1 + 2) \qquad 6\ (= 1 + 2 + 3) \qquad 10\ (= 1 + 2 + 3 + 4)$



(a) Draw pictures using dots to show that

$$\underbrace{3^2}_{\text{square}} = 2\underbrace{(1+2)}_{\text{triangular}} + 3 \text{ and } \quad \underbrace{4^2}_{\text{square}} = 2\underbrace{(1+2+3)}_{\text{triangular}} + 4.$$

Answer.



□

(b) Draw a picture using dots to show that

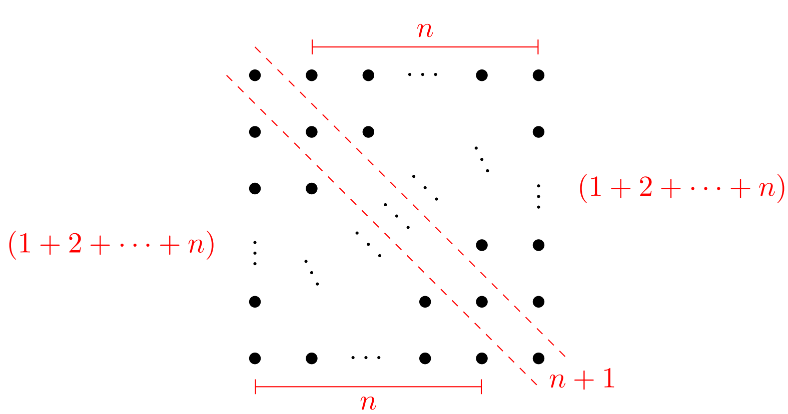
$$\underbrace{(n+1)^2}_{\text{square}} = 2\underbrace{(1+2+\cdots+n)}_{\text{triangular}} + (n+1) \tag{1}$$

Deduce that

$$1 + 2 + \cdots + n = n(n+1)/2 \tag{2}$$

(i.e. solve equation (1) for the triangular number).

Answer. We can see that $\underbrace{(n+1)^2}_{\text{square}} = 2\underbrace{(1+2+\cdots+n)}_{\text{triangular}} + (n+1)$ via the picture



Then solving for the triangular number, we get

$$\begin{aligned} 1 + 2 + \cdots + n &= \frac{1}{2}\left((n+1)^2 - (n+1)\right) \\ &= \frac{1}{2}(n+1)\left((n+1) - 1\right) \\ &= n(n+1)/2, \end{aligned}$$

as desired. □

(c) The number 1 is both square and triangular. Are there more?

(i) Do some examples by hand, and then try using a computer to generate more examples (generate the first 500 or so triangular numbers and their square roots using whatever your favorite computational program is—if you don't have one, try using a spreadsheet).

Answer. See spreadsheet for HW1. First three:

$$\begin{aligned} 36 &= 6^2 = 1 + 2 + \cdots 8, \\ 1225 &= 35^2 = 1 + 2 + \cdots + 49, \\ 41616 &= 204^2 = 1 + 2 + \cdots + 288. \end{aligned}$$

□

(ii) Consider equation (2): what can you say about the factors of n and $n+1$ that make it possible for $n(n+1)/2$ to be a perfect square?

Answer. We will prove some of these statements in the future, but for now, you should be able to do some examples to convince yourself of the following: If n is a multiple of a prime $p > 1$, then $n+1$ is not, and vice versa. So if $n(n+1)/2$ is a perfect square, then either

$$n \text{ and } (n+1)/2 \text{ are both perfect squares, i.e. } n = k^2 \text{ and } n+1 = 2\ell^2,$$

for some $k, \ell \in \mathbb{Z}$, or

$$n/2 \text{ and } n+1 \text{ are both perfect squares, i.e. } n = 2k^2 \text{ and } n+1 = \ell^2,$$

for some $k, \ell \in \mathbb{Z}$. So we're looking for integers k and ℓ such that

$$k^2 = 2\ell^2 - 1 \qquad \text{or} \qquad \ell^2 = 2k^2 + 1.$$

□

(iii) Make a hypothesis: do you think there are finitely many or infinitely many numbers that are both triangular and square?

Answer. From the last part, we can see that we can find these square-triangular numbers by finding integers k and ℓ such that

$$k^2 = 2\ell^2 - 1 \qquad \text{or} \qquad \ell^2 = 2k^2 + 1.$$

Therefore we have reduced the question to whether or not there are infinite integers m such that $2m^2 \pm 1$ is a perfect square (where m is either k or ℓ depending on the case, but since we only care if examples *exist*, we don't really care which is which). See the spreadsheet for homework 1 for several examples. This all serves to produce evidence that, **yes**, there may be infinitely many square numbers that are also triangular numbers.

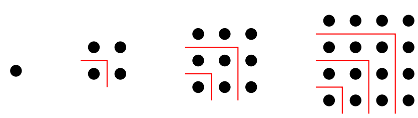
□

(d) Try adding up the first few odd numbers and see if the numbers you get satisfy some sort of pattern. Once you find the pattern, express it as a formula. Give a geometric verification (picture using dots) that your formula is correct. (Like in parts (a) and (b)).

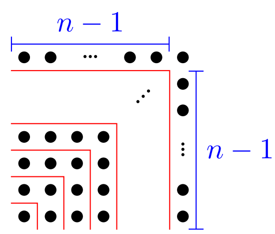
Answer. For the first few examples, we have

$$\begin{aligned} 1 &= 1, \\ 1+3 &= 4, \\ 1+3+5 &= 9, \\ 1+3+5+7 &= 16, \end{aligned}$$

and so on. These appear to be the first four square numbers. We can see why this is true by picturing, starting with a single dot, and adding strips of dots to the northeast:



In general, we can subdivide the $n \times n$ square into strips as follows:



Note that the largest strip has $(n-1)+(n-1)+1 = 2n-1$ dots. Further, it's two dots longer than the previous strip, which is two dots bigger than the previous, and so on. By induction, we see that our decomposition works in general, giving

$$1+3+5+\cdots+(2n-1) = n^2.$$

□

Exercise 3. Twin primes.
The *twin primes* are the prime numbers p such that $p+2$ or $p-2$ is also a prime:

$$\boxed{3,5,7}, \boxed{11,13}, \ldots.$$

(a) List all twin primes under 1000 (see the table of primes under 3000 on the last page). What is the first odd prime that is not a twin prime?

Answer. The twin primes under 1000 are

$3,5,7,$ $11,13,$ $17,19,$ $29,31,$ $41,43,$ $59,61,$ $71,73,$ $101,103,$ $107,109,$ $137,139,$ $149,151,$ $179,181,$ $191,193,$ $197,199,$ $227,229,$ $239,241,$ $269,271,$ $281,283,$ $311,313,$ $347,349,$ $419,421,$ $431,433,$ $461,463,$ $521,523,$ $569,571,$ $599,601,$ $617,619,$ $641,643,$ $659,661,$ $809,811,$ $821,823,$ $827,829,$ $857,859,$ $881,$ and $883.$

The first odd prime that is not a twin prime is 23.

□

(b) Why *don't* we have a name for a prime number p such that $p + 1$ or $p - 1$ is also a prime?

Answer. There is only one example! If p is odd, then $p \pm 1$ are even, and vice versa. Since there is only one even prime, 2, the only example of primes that differ by 1 is 2 and 3. $\square$

(c) Can you find any prime triplets, i.e. numbers p, $p + 2$, and $p + 4$ that are all prime? Are there are finitely many or infinitely many of these? (Look at your twin prime examples from part (a), and consider the odd numbers that come immediately before and after each pair of twin primes.)

Answer. Again, there is only one example: 3, 5, 7. Otherwise, you an check that of any three numbers consecutively differing by 2, say n, $n + 2$, and $n + 4$, one of them is a multiple of 3. The only multiple of 3 that is also prime is 3 itself. $\square$

(d) Do you think that there are infinitely many twin prime numbers? Do an internet search for "twin prime conjecture". Briefly, what is the current status of mathematic's progress on this question?

Answer. The current conjecture is that there *are* infinitely many twin primes, but this problem is still largely open. See articles on "twin prime conjecture". $\square$

Exercise 4. Primes of the form. . . It is generally believed that there are infinitely many primes of the form $n^2 + 1$, though this is still an open question and work is being done all the time. For example, the first three are

$$5 = 2^2 + 1, \quad 17 = 4^2 + 1, \quad \text{and} \quad 37 = 6^2 + 1.$$

(a) By computing values of $n^2 + 1$, and observing which are prime, list the next 5 primes of this form. (You can discount odd values of n—why?)

Answer. See HW1 spreadsheet for more examples. The next 5 are

$$101 = 10^2 + 1, \quad 197 = 14^2 + 1, \quad 257 = 16^2 + 1, \quad 401 = 20^1 + 1, \quad 577 = 24^2 + 1.$$

$\square$

(b) For what a do you think there might be infinitely many primes of the form $n^2 - a$? To answer this, do the following:

(i) Generate data for

$$n^2 - 1, \quad n^2 - 2, \quad n^2 - 3, \quad \text{and} \quad n^2 - 4,$$

for $n = 1, 2, \ldots, 10$, factoring composite numbers into primes (use a computer to assist generating your data). Report your findings in a table of values.

Answer.

n	n^2-1	n^2-2	n^2-3	n^2-4
2	3	2	1	0
3	$8=2^3$	$\mathbf{7}$	$6=2\cdot 3$	$\mathbf{5}$
4	$15=3*5$	$14=2*7$	$\mathbf{13}$	$12=2^2*3$
5	$24=2^3*3$	$\mathbf{23}$	$22=2*11$	$21=3*7$
6	$35=5*7$	$34=2*17$	$33=3*11$	$32=2^5$
7	$48=2^4*3$	$\mathbf{47}$	$46=2*23$	$45=3^2*5$
8	$63=3^2*7$	$62=2*31$	$\mathbf{61}$	$60=2^2*3*5$
9	$80=2^4*5$	$\mathbf{79}$	$78=2*3*13$	$77=7*11$
10	$99=3^2*11$	$98=2*6^2$	$\mathbf{97}$	$96=2^5*3$

□

(ii) For which values does n^2-a almost always factor?

Answer. If a is a perfect square, i.e. $a=k^2$ for some $k\in\mathbb{Z}_{>0}$, then

$$n^2-a=n^2-k^2=(n-k)(n+k).$$

So n^2-a factors unless $n-k=1$ (for example, if $a=1$, then $n=2$ gives a prime). □

(iii) Make a hypothesis.

Answer. There seem to be lots of primes of the form n^2-a when a is not a perfect square. □

Exercise 5. Primitive Pythagorean triples.

(a) At the end of Chapter 2, the book provides several examples of PPTs. Use a computer to generate 20 *more* examples. (Be sure to enter some of the same values of s and t into your table as given on p. 18, though, to check that your formulas are working properly.) In your homework writeup, report 5 of them in a similar table.

Answer.

s	t	$a=st$	$b=(s^2-t^2)/2$	$c=(s^2+t^2)/2$
11	9	99	20	101
11	7	77	36	85
11	5	55	48	73
11	3	33	56	65
11	1	11	60	61

□

(b) Looking at all the combined data, do you see any more of a pattern of which kinds of numbers a or b can be? (For example, can a be *any* odd number? Or are there restrictions? Can b be *any* even number?) Make hypotheses; prove them if you can.

Answer. Any odd number can appear as the a in a primitive Pythagorean triple. To find such a triple, we can just take $t=a$ and $s=1$ in the Pythagorean Triples Theorem. This gives the primitive Pythagorean triple $(a,(a^2?1)/2,(a^2+1)/2)$.

Looking at the table, it seems first that b must be a multiple of 4, and second that every multiple of 4 seems to be possible. We know that b looks like $b = (s^2?t^2)/2$ with s and t odd. This means we can write $s = 2m + 1$ and $t = 2n + 1$. Multiplying things out gives

$$b = \frac{(2m+1)^2 - (2n+1)^2}{2} = 2m^2 + 2m - 2n^2 - 2n = 2m(m+1) - 2n(n+1).$$

Can you see that $m(m + 1)$ and $n(n + 1)$ must both be even, regardless of the value of m and n? So b must be divisible by 4.

On the other hand, if b is divisible by 4, then we can write it as $b = 2^r B$ for some odd number B and some $r \geq 2$. Then we can try to find values of s and t such that $(s^2 - t^2)/2 = b$. We factor this as

$$(s-t)(s+t) = 2b = 2^{r+1}B.$$

Now both $s - t$ and $s + t$ must be even (since s and t are odd), so we might try

$$s - t = 2r \quad \text{and} \quad s + t = 2B.$$

Solving for s and t gives $s = 2^{r-1} + B$ and $t =?2^{r?1} + B$. Notice that s and t are odd, since B is odd and $r \geq 2$. Then

$$a = st = B^2 - 2^{2r-2},$$

$$b = \frac{s^2 - t^2}{2} = 2^r B,$$

$$c = \frac{s^2 + t^2}{2} = B^2 + 2^{2r-2}.$$

This gives a primitive Pythagorean triple with the right value of b provided that $B > 2^{r-1}$. On the other hand, if $B < 2^{r-1}$, then we can just take $a = 2^{2r-2} - B^2$ instead.

□

(c) We showed that in any primitive Pythagorean triple (a, b, c), either a or b is even. Use the same sort of argument to show that either a or b must be a multiple of 3.

Proof. If a is not a multiple of 3, it must equal either $3x + 1$ or $3x + 2$ for some $x \in \mathbb{Z}_{\geq 0}$. Similarly, if b is not a multiple of 3, it must equal $3y + 1$ or $3y + 2$ for some $y \in \mathbb{Z}_{\geq 0}$. There are then four possibilities for $a^2 + b^2$, namely

$$\begin{aligned}
a^2 + b^2 &= (3x+1)^2 + (3y+1)^2 \\
&= 9x^2 + 6x + 1 + 9y^2 + 6y + 1 \\
&= 3(3x^2 + 2x + 3y^2 + 2y) + 2, \\
a^2 + b^2 &= (3x+1)^2 + (3y+2)^2 \\
&= 9x^2 + 6x + 1 + 9y^2 + 12y + 4 \\
&= 3(3x^2 + 2x + 3y^2 + 4y + 1) + 2, \\
a^2 + b^2 &= (3x+2)^2 + (3y+1)^2 \\
&= 9x^2 + 12x + 4 + 9y^2 + 6y + 1 \\
&= 3(3x^2 + 4x + 3y^2 + 2y + 1) + 2, \\
a^2 + b^2 &= (3x+2)^2 + (3y+2)^2 \\
&= 9x^2 + 12x + 4 + 9y^2 + 12y + 4 \\
&= 3(3x^2 + 4x + 3y^2 + 4y + 2) + 2.
\end{aligned}$$

So if a and b are not multiples of 3, then $c^2 = a^2 + b^2$ is always exactly 2 more than a multiple of 3. But regardless of whether c is $3z$ or $3z+1$ or $3z+2$, the numbers c^2 cannot be 2 more than a multiple of 3:

$$(3z)^2 = 3(3z^2),$$

$$(3z+1)^2 = 3(3z^2 + 2z) + 1,$$

$$(3z+2)^2 = 3(3z^2 + 4z + 1) + 1.$$

Therefore at least one of a or b *must* be a multiple of 3.

□

(d) Both $(33, 56, 65)$ and $(63, 16, 65)$ are PPTs.

(i) Compute s and t values for these two PPTs.

Answer. For $(33, 56, 65)$,

$$s = \sqrt{c+b} = \sqrt{65+56} = \sqrt{121} = 11,$$

and

$$t = \sqrt{c-b} = \sqrt{65-56} = \sqrt{9} = 3.$$

For $(63, 16, 65)$,

$$s = \sqrt{c+b} = \sqrt{65+16} = \sqrt{81} = 9,$$

and

$$t = \sqrt{c-b} = \sqrt{65-16} = \sqrt{49} = 7.$$

□

(ii) There are also two PPTs with $c = 85$. What are they?

Answer. We need b's where $85 + b$ and $85 - b$ are both perfect squares. The two solutions are $b = 84$ (so that $a = 13$) and $b = 36$ (so that $a = 77$).

Note: To find these and the examples in the next couple of parts, I set up a spreadsheet calculation using the fact that whatever b is,

$$c + b = s^2 \quad \text{and} \quad c - b = t^2$$

must be perfect squares. So I did my calculation plugging in all possible s-values, then calculating $b = s^2 - c$, so that

$$t = \sqrt{c - (s^2 - c)} = \sqrt{2c - s^2}$$

and looking for examples where the resulting t is an integer. Then I used s and t to calculate the corresponding a and b values, and threw in a column for verifying that $a^2 + b^2 - c^2 = 0$, for good measure. Note that in my equations, I stuck the c-value in cell G2, and then I used "G2" so that later I could fiddle with that number to do lots of examples. The $ signs "protect" that value so that as I past the equations into lower rows, they don't adjust that variable name accordingly. The bigger my example, though, the more rows I needed. You're seeing what the spreadsheet looks like after I deleted the extra rows (where $s > \sqrt{c}$).

□

(iii) Give another example of a pair of PPTs that have the same c value.

Answer. Another example is with $c = 221 = 13*17$, with $b = 220$ and $a = 21$, and $b = 140$ and $a = 171$. (See spreadsheet.) □

Answer. □

(iv) Can you find a value of c for which there are *three* primitive Pythagorean triples with the same c? More?
[Hint: consider the prime factorizations of 65 and 85.]

Answer. A general rule is that if $c = p_1 p_2 \cdots p_r$ is a product of r distinct odd primes that all leave a remainder of 1 when divided by 4, then c appears as the hypotenuse in 2^{r-1} primitive Pythagorean triples. (This is counting (a, b, c) and (b, a, c) as the same triple.) So for example, $c = 5 * 13 * 17 = 1105$ appears in 4 triples,

$$1105^2 = 576^2 + 943^2 = 744^2 + 817^2 = 264^2 + 1073^2 = 47^2 + 1104^2.$$

Don't worry if you didn't get here though! At this point, it would be pretty difficult to prove the general rule using only the material we have developed so far. □

(e) A nonzero integer d is said to *divide* an integer m if $m = dk$ for some $k \in \mathbb{Z}$. Show that if d divides both m and n, then d also divides $m - n$ and $m + n$.

Proof. Suppose d divides both m and n, so that

$$m = dk \quad \text{and} \quad n = d\ell,$$

for some $k, \ell \in \mathbb{Z}$. Then

$$m - n = dk - d\ell = d(k - \ell)$$

and

$$m + n = dk + d\ell = d(k + \ell).$$

So since both $k - \ell$ and $k + \ell$ are integers, d also divides $m - n$ and $m + n$. $\square$

(f) Do an internet search for "Plimpton 322". What is it? What do we know about it? What is the recent news about our understanding of it (recent as in the last week or so)? Write a brief summary, citing your sources (*don't just rely on Wikipedia*).