Review: relations

A binary relation on a set A is a subset $R \subseteq A \times A$, where elements (a, b) are written as $a \sim b$.

Example: $A = \mathbb{Z}$ and $R = \{a \sim b \mid a \equiv b \pmod{n}\}$.

A binary relation on a set A is...

(R) reflexive if $a \sim a$ for all $a \in A$;

(S) symmetric if $a \sim b$ implies $b \sim a$;

(T) transitive if $a \sim b$ and $b \sim c$ implies $a \sim c$, i.e.

$$(a \sim b \wedge b \sim c) \Rightarrow a \sim c$$

An equivalence relation on a set A is a binary relation that is reflexive, symmetric, *and* transitive.

Review: set theoretic definition of the numbers.

Natural numbers:

Let $0 = \varnothing$.

Given n, define the successor to n as $S(n) = n \cup \{n\}$.

(By "successor to n" we basically mean $n + 1$.)

Let $\mathbb{Z}_{\geqslant 0}$ be the set of all sets generated by 0 and S.

Integers:

Define $\mathbb{Z}$ by formally letting

$$-\mathbb{Z}_{\geqslant 0} = \{-n \mid n \in \mathbb{Z}_{\geqslant 0}\}, \quad \text{where } -0 = 0;$$

and $\mathbb{Z} = \mathbb{Z}_{\geqslant 0} \cup -\mathbb{Z}_{\geqslant 0}$. Extend $S : \mathbb{Z} \to \mathbb{Z}$ by defining $S(-a)$ for any $-a \in -\mathbb{N} - \{0\}$ as

$$S(-a) = -b, \quad \text{where } S(b) = a.$$

Some operations:

- Define $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by, for all $a, b \in \mathbb{N}$, by

$$a + 0 = a \qquad \text{and} \qquad a + S(b) = S(a + b).$$

- Define $\cdot : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by, for all $a, b \in \mathbb{N}$,

$$n \cdot 0 = 0 \qquad \text{and} \qquad a \cdot S(b) = (a \cdot b) + a.$$

Review:

Some properties of $+$ and $\cdot$ (we present without proof):

1. Addition and multiplication satisfy commutativity, associativity, and distributivity.
2. We still have $a + 0 = a = 0 + a$ (additive identity) and $a \cdot 1 = a = 1 \cdot a$ (multiplicative identity) for all $a \in \mathbb{Z}$.
3. We also have $a + (-a) = 0$ (prove). (additive inverses)

We call any number system that has an addition and multiplication that satisfy all these properties a (commutative) ring.

Order: For $a, b \in \mathbb{Z}$, we say $a \leqslant b$ if $b = S(S(\cdots S(a)\cdots))$.

Properties of order (we present without proof):

(i) For all $a, b \in \mathbb{N}$, we have $a \leqslant b$ or $b \leqslant a$.

(ii) If $a \leqslant b$ and $b \leqslant a$, then $a = b$.

(iii) If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

(iv) If $a \leqslant b$ then $a + c \leqslant b + c$.

(v) If $a \leqslant b$ then $a \cdot c \leqslant b \cdot c$.

Rational numbers

Let

$$Q = \mathbb{Z} \times (\mathbb{Z} - \{0\}),$$

and define an equivalence relation on Q by

$$(a, b) \sim (x \cdot a, x \cdot b) \quad \text{for all} \quad x \in \mathbb{Z} - \{0\}.$$

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Under this equivalence relation, write

$$\frac{a}{b} = [(a, b)].$$

Then rational numbers are

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

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$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}.$$

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2. We still have $x + 0 = x$ (additive identity) and $x \cdot 1 = x$ (multiplicative identity) for all $x \in \mathbb{Q}$.
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In addition, for all $a/b \in \mathbb{Q}$ with $a \neq 0$,

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This makes $\mathbb{Q}$ a field (again, modern algebra).

Order on $\mathbb{Q}$

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1. $\frac{a}{b} \leqslant \frac{c}{d}$ whenever $a \cdot d \leqslant b \cdot c$;
2. $-\frac{a}{b} \leqslant \frac{c}{d}$; and
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Then, again,

(i) For all $a, b \in \mathbb{N}$, we have $a \leqslant b$ or $b \leqslant a$.

(ii) If $a \leqslant b$ and $b \leqslant a$, then $a = b$.

(iii) If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

(iv) If $a \leqslant b$ then $a + c \leqslant b + c$.

(v) If $a \leqslant b$ and $0 \leqslant c$, then $ac \leqslant bc$.

This makes $\mathbb{Q}$ into an ordered field.

Let X be an ordered set of numbers (think $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and, eventually, $\mathbb{R}$), and let S be a nonempty subset of X.

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Careful writing note: Compare to, "The *maximum* of S is an element $x \in S$ satisfying $y \leqslant x$ for all $y \in S$.

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Note: Min/max don't depend on the set X!

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(b) Similarly, if there exists $\ell \in X$ such that $s \geqslant \ell$ for all $s \in S$, then ℓ is called a lower bound of S and the set S is said to be bounded below (by ℓ).

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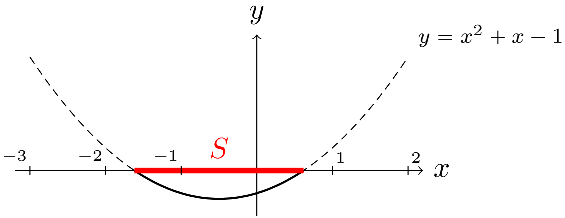
Note: Upper and lower bounds *do* depend on the set X.

Consider $X = \mathbb{Q}$ and

$$S = \{x \in \mathbb{Q} \mid x^2 + x - 1 \leqslant 0\}$$



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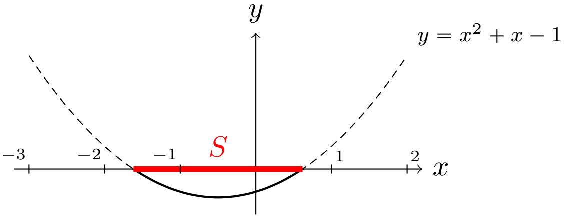

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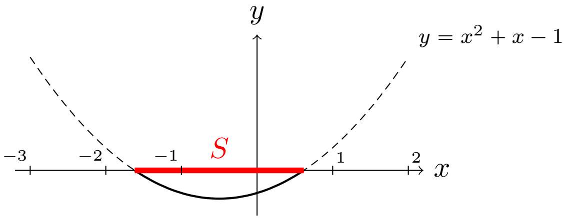

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But S *is* bounded above and below,

e.g. $u = 1$ and $\ell = -2$;

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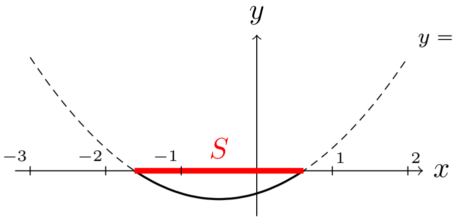

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So S doesn't have a minimum or a maximum."

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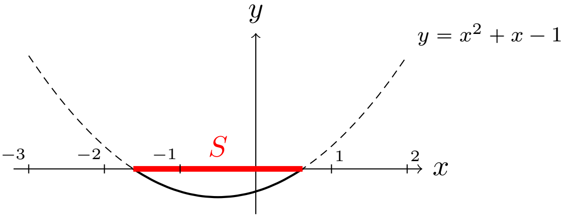

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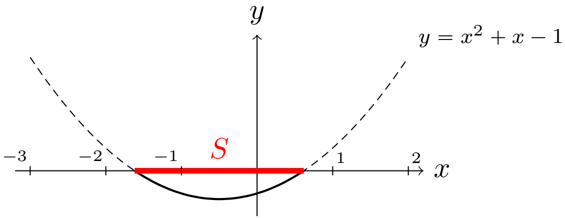

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$\ell = \frac{1}{2}(-1 - \sqrt{5})$ *and* $u = \frac{1}{2}(-1 + \sqrt{5})$."

Let X be an ordered set of numbers (think $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and, eventually, $\mathbb{R}$), and let S be a nonempty subset of X.

From before: If there exists $u \in X$ such that $s \leqslant u$ for all $s \in S$, then u is called an upper bound of S and the set S is said to be bounded above (by u). Similarly, a lower bound is a number $\ell \in X$ such that $s \geqslant \ell$ for all $s \in S$; if ℓ exists, we say S is bounded below.

(a) If S is bounded above, we call an upper bound U satisfying

$$U \leqslant u \quad \text{for all upper bounds } u$$

the least upper bound or supremum of S, denoted by $\sup S$.

$$\sup S = \min\left(\{u \in X \mid u \geqslant s \text{ for all } s \in S\}\right)$$

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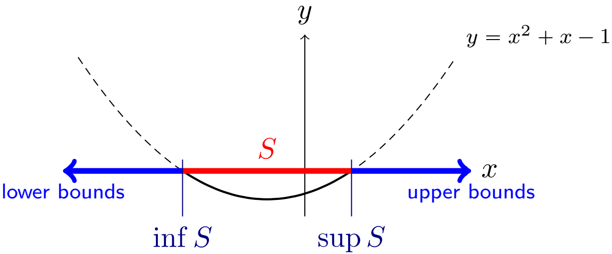

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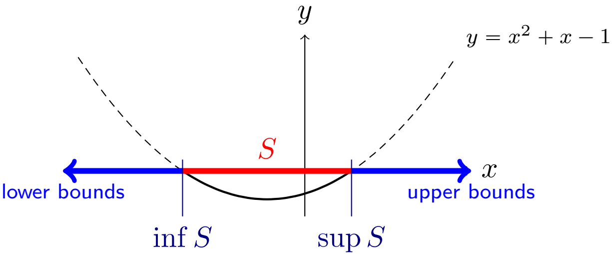

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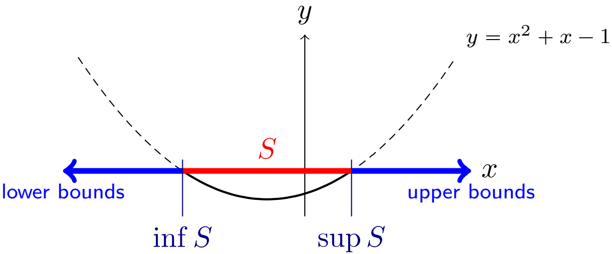

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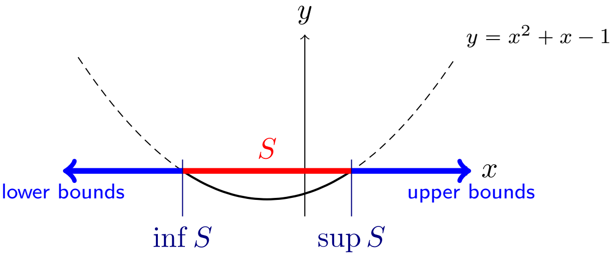

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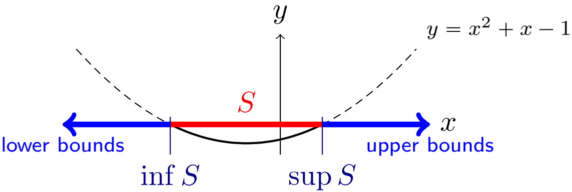

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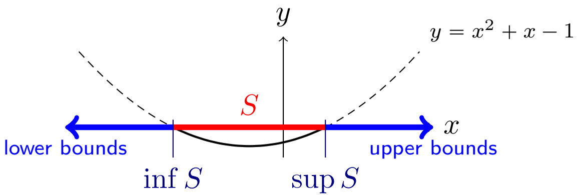

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Completeness Axiom: Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound, i.e. for all $S \subseteq \mathbb{R}$, if S is bounded above, then $\sup S$ exists and is in $\mathbb{R}$.

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Let $\mathcal{R}$ be the set of subsets of $\mathbb{Q}$ that satisfy the following:
$C \in \mathcal{R}$ whenever

1. $C \subsetneq \mathbb{Q}$ and $C \neq \varnothing$ (C is a proper, non-empty subset of $\mathbb{Q}$);
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Thm. The completeness axiom holds.

(This *should* feel uncomfortable. . . an axiom shouldn't have to be proven!
Whether this is an axiom or a theorem depends on your perspective. . .)

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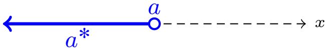

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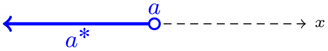

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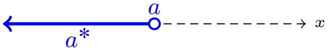

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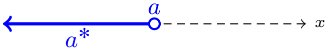

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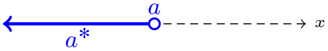

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Instead: for non-negative $\alpha, \beta \in \mathcal{R}$ (i.e. $\alpha_{\geqslant 0}, \beta_{\geqslant 0} \neq \varnothing$), define

1. $\alpha \cdot \beta = \{a \cdot b \mid a \in \alpha_{\geqslant 0}, b \in \beta_{\geqslant 0}\} \cup \mathbb{Q}_{<0}$;
2. $-\alpha = \{-x \in \mathbb{Q} \mid x > a \text{ for all } a \in \alpha\}$;
3. $-\alpha \cdot \beta = -(\alpha \cdot \beta)$; and
4. $(-\alpha) \cdot (-\beta) = \alpha \cdot \beta$.

Addition: For $\alpha, \beta \in \mathcal{R}$, define

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Again. . .

1. Addition and multiplication still satisfy commutativity, associativity, and distributivity.
2. We still have $\alpha + 0^* = x$ (additive identity) and $\alpha \cdot 1^* = \alpha$ (multiplicative identity) for all $\alpha \in \mathcal{R}$.
3. We also have that $\alpha + (-\alpha) = 0^*$. (additive inverses)
4. Dor all $\alpha \in \mathcal{R}$ with $\alpha \neq 0^*$, there exists $\alpha^{-1} \in \mathcal{R}$ that satisfies
$$\alpha \cdot \alpha^{-1} = 1^* \quad \text{(multiplicative inverses)}.$$

So $\mathcal{R}$ a field (again, modern algebra).

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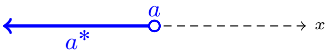

For all $a \in \mathbb{Q}$, we can concretely identify a with $a^* = \{x \in \mathbb{Q} \mid x < a\} \in \mathcal{R}$. Namely,

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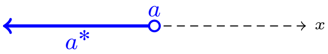

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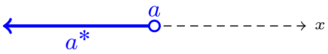

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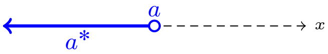

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4. **Revise, revise, revise.**
   When solving math problems, a lot goes on behind the scenes. Don't be afraid to write down logical fallacies (like starting with the conclusion, or "proof by example") in the privacy of your own home. Just don't *stop* there!

   Tip: Keep old ideas or notes to yourself in your .tex file, commented out with %'s, in case you need them again later.

Wrapping up

1. **Context is king!**
2. **Go slowly when reading/writing new math.**
3. **Always do examples!**

   Illustrative examples can help you understand the big picture; extreme examples help you understand the details. And if you don't know where to start, then do an example!
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   While a logical argument needs to come in a logical order, there isn't just one good order to explain all of math. In particular, every good answer spins off many good questions!

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6. **You can do it!!**
   Be kind to yourself! If you don't get something right away, that doesn't mean you're stupid, or that you can't get there. Math is hard, but doable; and the struggle is what makes the breakthroughs so fun!