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Is $\sim$ is an equivalence relation?

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Yes! This is an equivalence relation!

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So no, it is not an equivalence relation.

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Read: Why reflexivity doesn't follow from symmetry and transitivity.

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Example: Consider the equivalence relation on $A = \{a, b, c\}$ given by

$$a \sim a, \quad b \sim b, \quad c \sim c, \quad a \sim c, \quad \text{and} \quad c \sim a.$$

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Then

$$[a] = \{a, c\} = [c], \quad \text{and}$$
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are the two equivalence classes in A (with respect to this relation).

(We say there are two, *not three*, since "the equivalence classes" refers to the sets themselves, not to the elements that generate them.)

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Example: We showed that

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In general, if $x \in [y]$, that means $y \sim x$.

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Claim: $x \in [y]$ if and only if $[x] = [y]$.

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Claim: $x \in [y]$ if and only if $[x] = [y]$.

We call any element a of a class C representative of C (since we can write $C = [a]$ for any $a \in C$).

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So $\boxed{\mathbb{Z} = [0] \sqcup [1] \sqcup [2] \sqcup [3] \sqcup [4]}$.

Ok, so what **are** numbers, anyway?

Recall from the homework, the Zermelo-Fraenkel axioms of set theory, which tells us how to compare sets, put sets in other sets, and to take unions, intersection, and power sets of sets. ✓

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Set theoretic definition of the natural numbers. ($\mathbb{Z}_{\geqslant 0}$)
Let $0 = \varnothing$.
Given n, define the successor to n as $S(n) = n \cup \{n\}$.

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(By "successor to n" we basically mean $n + 1$.)
Let $\mathbb{N}$ be the set of all sets generated by 0 and S.
For example,

$$1 = 0 \cup \{0\} = \varnothing \cup \{\varnothing\} = \{\varnothing\},$$

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Ok, so what **are** numbers, anyway?

Recall from the homework, the Zermelo-Fraenkel axioms of set theory, which tells us how to compare sets, put sets in other sets, and to take unions, intersection, and power sets of sets. ✓

Set theoretic definition of the natural numbers. ($\mathbb{Z}_{\geqslant 0}$)
Let $0 = \varnothing$.
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and so on. (Note that we identified n with $|n|$.) Compute 4.
Think: Given $n, m \in \mathbb{N}$, are $n \cup m$ and/or $n \cap m$ elements of $\mathbb{N}$? If so, what elements are they?

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For example,

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Check that $a + 3 = S(S(S(a))) = S^3(a)$.

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Check that $a + 3 = a + a + a$.

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Check that $a + 3 = a + a + a$. Think: $a \cdot b = S^{ab}(0)$.

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Think: $a \cdot b = S^{ab}(0)$.

Properties:

1. Addition is commutative, i.e. $a + b = b + a$ for all $a, b \in \mathbb{N}$.
2. Addition is associative, i.e. $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{N}$.
3. Multiplication is commutative, i.e. $a \cdot b = b \cdot a$ for all $a, b \in \mathbb{N}$.
4. Multiplication is associative, i.e. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{N}$.
5. Multiplication is distributive across addition, i.e. $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in \mathbb{N}$.

(These all follow from the definitions, but we'll skip proofs for the sake of time.)

Peano axioms

The natural numbers $\mathbb{N}$ are defined by the following axioms.

1. We have $0 \in \mathbb{N}$. (technically, $0 = \varnothing$)
2. There exists an a successor function $S : \mathbb{N} \to \mathbb{N}$ (namely, if $n \in \mathbb{Z}$, then $S(n) \in \mathbb{N}$) satisfying
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(b) By changing 0 out for something else (like 1), or changing $S(n)$ to something else (like $n - 1$), we can generate other sets that are basically the natural numbers all over again. This is why we're not fussy about whether $\mathbb{N}$ is $\mathbb{Z}_{\geqslant 0}$ or $\mathbb{Z}_{>0}$.

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(c) The last axiom (1(iii)) is the basis of proof by induction.

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Integers

Now that we have $\mathbb{N}$, we can define $\mathbb{Z}$ by formally letting

$$-\mathbb{N} = \{-n \mid n \in \mathbb{N}\}, \quad \text{where } -0 = 0;$$

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Properties:

1. Addition and multiplication still satisfy commutativity, associativity, and distributivity.
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We call any number system that has an addition and multiplication that satisfy all these properties a ring (modern algebra).

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We can also extend our definition of order to $\mathbb{Z}$. The only modification is:

(i) For all $a, b \in \mathbb{N}$, we have $a \leqslant b$ or $b \leqslant a$.

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These properties make $\mathbb{Z}$ an ordered ring.

Rational numbers

Let

$$Q = \mathbb{Z} \times (\mathbb{Z} - \{0\}),$$

and define an equivalence relation on Q by

$$(a, b) \sim (x \cdot a, x \cdot b) \quad \text{for all} \quad x \in \mathbb{Z} - \{0\}.$$

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Under this equivalence relation, write

$$\frac{a}{b} = [(a, b)].$$

Then rational numbers are

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

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This makes $\mathbb{Q}$ a field (again, modern algebra).

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This makes $\mathbb{Q}$ into an ordered field.