Functions

Let $f : X \to Y$ be a function. Recall, the image of f is

$$f(X) = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}.$$

Further, f is...

- injective if at most one $x \in X$ maps to each $y \in Y$, i.e.

  $$\text{if } f(x_1) = f(x_2) \text{ then } x_1 = x_2.$$

  Ex. $f : \mathbb{R}_{>0} \to \mathbb{R}$ defined by $x \mapsto x^2$.

  Non-ex. $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$.

- surjective if every $y \in Y$ gets mapped to, i.e.

  $$\text{for all } y \in Y, \text{ there exists } x \in X \text{ such that } f(x) = y.$$

  Ex. $f : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ defined by $x \mapsto x^2$.

  Non-ex. $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$.

- bijective if it's both injective and surjective.

  Ex. $f : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ defined by $x \mapsto x^2$.

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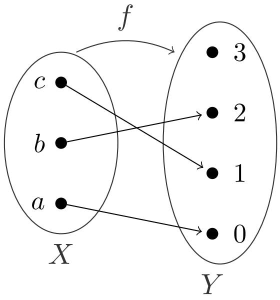
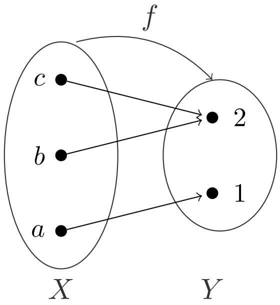
Injective, not surjective:



Surjective, not injective:



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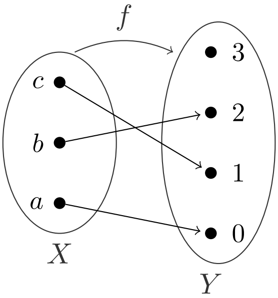

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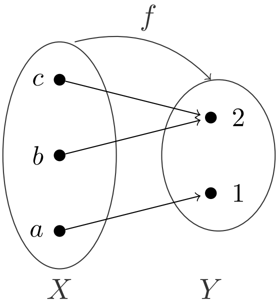

To show $f : X \to Y$ is injective:

Start: *Suppose* $f(x_1) = f(x_2) \ldots$ Goal: *Show* $x_1 = x_2$.

To show $f : X \to Y$ is surjective:

Start: *Let* $y \in Y \ldots$ Goal: *Find* $x \in X$ *such that* $f(x) = y$.

For a function $f : X \to Y$, and an element $y \in Y$, let

$$f^{-1}(y) = \{x \in X \mid f(x) = y\} \subset X,$$

called the inverse image or preimage of y. Note that this is abusing notation: we write f^{-1} whether or not f is invertible; and $f^{-1}(y)$ is a *set*, not an *element*.

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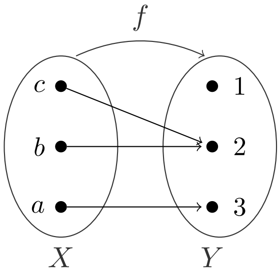

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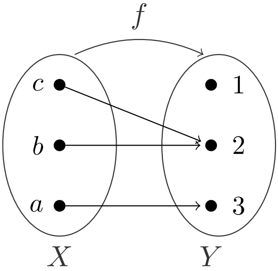

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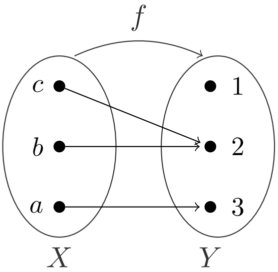

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Thm. For nonempty sets X and Y, a function $f : X \to Y$ is invertible if and only if it is bijective.

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Proof.

Suppose f is bijective. Since f is surjective, for all $y \in Y$, we have $|f^{-1}(y)| \geqslant 1$. And since f is injective, for any $x_1, x_2 \in f^{-1}(y)$, we have $x_1 = x_2$. So $|f^{-1}(y)| \leqslant 1$. Therefore, for all $y \in Y$, we have $|f^{-1}(y)| = 1$, so that f is invertible.

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Now suppose f is invertible. Thus for all $y \in Y$, we have $|f^{-1}(y)| = 1$. Therefore, for all $y \in Y$, $f(y) \neq \varnothing$, so that f is surjective. And for all $y \in Y$, since $f^{-1}(y)$ has exactly one element, it has at most one element. So f is injective. Therefore, f is bijective.

$\square$

Cardinality of sets

Two sets A and B have the same size, or cardinality, if there is a bijection $f : A \to B$.

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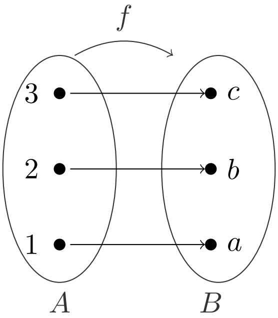

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(This allows us to measure the relative sizes of sets, even if they happen to be infinite!)

Example: The sets $\mathbb{Z}_{\geqslant 0}$ and $\mathbb{Z}_{>0}$ have the same cardinality since

$$\begin{array}{rcl} f : \mathbb{Z}_{>0} & \to & \mathbb{Z}_{\geqslant 0} \\ x & \mapsto & x - 1 \end{array}$$

is a bijective map.

Countably infinite sets

A set is countable if it is either finite or the same cardinality as the natural numbers ($\mathbb{N} = \mathbb{Z}_{>0}$). If a set A is not finite but is countable, we say A is "countably infinite" and write $|A| = \aleph_0$ (pronounced "aleph naught" or "aleph null").

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Countably infinite sets

A set is countable if it is either finite or the same cardinality as the natural numbers ($\mathbb{N} = \mathbb{Z}_{>0}$). If a set A is not finite but is countable, we say A is "countably infinite" and write $|A| = \aleph_0$ (pronounced "aleph naught" or "aleph null"). To show that $|A| = \aleph_0$: show A is not finite, and give a bijection $f : \mathbb{Z}_{\geqslant 0} \to A$.

Examples:

1. $\mathbb{Z}_{>0}$ is countably infinite:
   It is not finite, and $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ by $x \mapsto x$ is a bijection.
2. $\mathbb{Z}_{\geqslant 0}$ is countably infinite:
   It is not finite, and $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{\geqslant 0}$ by $x \mapsto x - 1$ is a bijection.
3. $\mathbb{Z}$ is countably infinite: Not finite, and $f : \mathbb{Z}_{>0} \to \mathbb{Z}$ by $x \mapsto (-1)^x \lfloor x/2 \rfloor$ is a bijection.

$$\begin{array}{rcccccccccc}
\mathbb{Z}_{>0}: & \cdots & 9 & 7 & 5 & 3 & 1 & 2 & 4 & 6 & 8 \cdots \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbb{Z}: & \cdots & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \quad \cdots
\end{array}$$

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\end{array}$$

This at least gives us a "list" of integers,

$$1 : 0, \quad 2 : 1, \quad 3 : -1, \quad 4 : 2, \quad 5 : -2, \quad \ldots .$$

If I know that every integer appears on this list somewhere, then I know that the integers are countable. (Ok answer)

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$$f : \mathbb{Z}_{>0} \to \mathbb{Z} \qquad x \mapsto \begin{cases} x/2 & \text{if } x \text{ is even,} \\ -(x-1)/2 & \text{if } x \text{ is odd.} \end{cases} \qquad \text{(Better answer)}$$

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To be even more sophisticated, we used the floor function to get a closed form answer:

$$\begin{cases} x/2 & \text{if } x \text{ is even,} \\ -(x-1)/2 & \text{if } x \text{ is odd} \end{cases} = (-1)^x \lfloor x/2 \rfloor,$$

for $x \in \mathbb{Z}_{>0}$. (Best answer)

Recall that $|A| = |B|$ if and only if there is a bijection $f : A \to B$.

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If we know that $|A| = \aleph_0$ and $f : A \to B$ is a bijection, then $|B| = |A| = \aleph_0$.

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Example: To show that $2\mathbb{Z} = \{$ even integers $\}$ is countably infinite, we could construct a bijection like in the previous example.

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$$\begin{aligned} f : \mathbb{Z} &\to 2\mathbb{Z} \\ x &\mapsto 2x \end{aligned} \quad \text{is a bijection,}$$

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so that $|2\mathbb{Z}| = |\mathbb{Z}| = \aleph_0$.

Examples: For each of the following, show that the set is countably infinite. (Define a bijective function to something that we know to be countably infinite if it's not too hard; otherwise, explain how to make the list.)

1. The set of negative integers ($\mathbb{Z}_{<0}$).
2. The set of integers less than 100 ($\mathbb{Z}_{<100}$).
3. The set of integers that are integer multiples of 3 ($3\mathbb{Z}$).
4. The set of integers that are not integer multiples of 3 ($\mathbb{Z} - 3\mathbb{Z}$).

The rational numbers

Claim: $\mathbb{Q}_{>0}$ is countably infinite.

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	1	2	3	4	5	$\cdots$
1	$1/1$	$2/1$	$3/1$	$4/1$	$5/1$	
2	$1/2$	$2/2$	$3/2$	$4/2$	$5/2$	
3	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	
4	$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	
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3	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	
4	$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	
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3	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	
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3	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	
4	$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	
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2	$\overset{3}{\boxed{1/2}}$	$2/2$	$3/2$	$4/2$	$5/2$	
3	$\overset{4}{\boxed{1/3}}$	$2/3$	$3/3$	$4/3$	$5/3$	
4	$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	
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3	$\overset{4}{\boxed{1/3}}$	$2/3$	$3/3$	$4/3$	$5/3$	
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3	$\overset{4}{\boxed{1/3}}$	$2/3$	$3/3$	$4/3$	$5/3$	
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2	$\overset{3}{\boxed{1/2}}$	$2/2$	$\overset{7}{\boxed{3/2}}$	$4/2$	$5/2$	
3	$\overset{4}{\boxed{1/3}}$	$\overset{8}{\boxed{2/3}}$	$3/3$	$4/3$	$5/3$	
4	$\overset{9}{\boxed{1/4}}$	$2/4$	$3/4$	$4/4$	$5/4$	
5	$\overset{10}{\boxed{1/5}}$	$2/5$	$3/5$	$4/5$	$5/5$	
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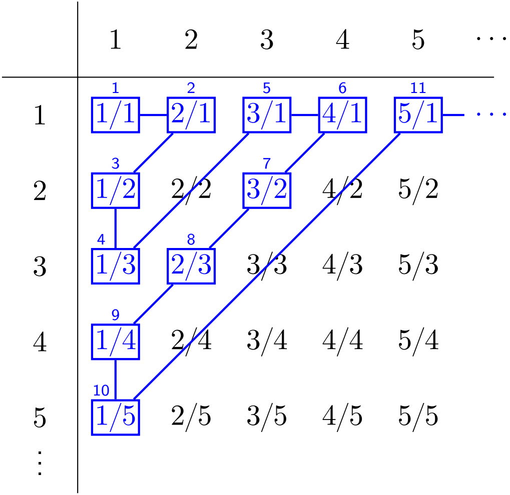

(skip prev. counted fractions)

Then use the same alternating map that we did for $\mathbb{Z}_{>0} \to \mathbb{Z}$ to build a bijection $\mathbb{Q}_{>0} \to \mathbb{Q}$, yielding a bijection $\mathbb{Z}_{>0} \to \mathbb{Q}_{>0} \to \mathbb{Q}$.

Are there sets that are *not* countable?

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Theorem. The set of real numbers in the interval $[0, 1)$ is not countable.

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Theorem. The set of real numbers in the interval $[0, 1)$ is not countable.

Proof outline: We will prove this by contradiction.
Suppose that the set $[0, 1)$ is countable, so that the real numbers in $[0, 1)$ can be listed.
Take one such list.

Goal: Show that the list isn't complete. Namely, algorithmically produce an element of $[0, 1)$ that isn't on any fixed list.

Take this is the supposedly complete list of real numbers in $[0, 1)$.

For example:

1. 0.001240191057...
2. 0.123451234512...
3. 0.333333333333...
4. 0.500000000000...
5. 0.121212121212...
6. 0.555555555555...
7. 0.141592653589...
8. 0.001850000000...
9. 0.111111111111...
10. 0.750000000000...
11. 0.948797362471...

$\vdots$

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Algorithm for producing a number that is not on the list:

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$\vdots$

Algorithm for producing a number that is not on the list:

In the ith number in the list, highlight the ith digit.

Take this is the supposedly complete list of real numbers in $[0, 1)$.

For example:

1. $0.\mathbf{0}01240191057\ldots$
2. $0.1\mathbf{2}3451234512\ldots$
3. $0.33\mathbf{3}333333333\ldots$
4. $0.500\mathbf{0}00000000\ldots$
5. $0.1212\mathbf{1}2121212\ldots$
6. $0.55555\mathbf{5}555555\ldots$
7. $0.141592\mathbf{6}53589\ldots$
8. $0.0018500\mathbf{0}0000\ldots$
9. $0.11111111\mathbf{1}111\ldots$
10. $0.750000000\mathbf{0}00\ldots$
11. $0.9487973624\mathbf{7}1\ldots$

$\vdots$

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11. 0.9487973624**7**1...

$\vdots$

Algorithm for producing a number that is not on the list:

In the ith number in the list, highlight the ith digit.

Build a new number as follows:

i. If the highlighted digit of the ith number is a 0, then make the corresponding digit of the new number a 1.

ii. If the highlighted digit of the ith number is not a 0, then make the corresponding digit of the new number a 0.

Take this is the supposedly complete list of real numbers in $[0,1)$.

For example:

1. 0.**0**01240191057...
2. 0.1**2**3451234512...
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Example:

0.1

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⋮

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i. If the highlighted digit of the ith number is a 0, then make the corresponding digit of the new number a 1.

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Example:

0.10

Take this is the supposedly complete list of real numbers in $[0,1)$.

For example:

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3. 0.33**3**333333333...
4. 0.500**0**00000000...
5. 0.1212**1**2121212...
6. 0.55555**5**555555...
7. 0.141592**6**53589...
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$\vdots$

Algorithm for producing a number that is not on the list:

In the ith number in the list, highlight the ith digit.

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Example:

0.100

Take this is the supposedly complete list of real numbers in $[0, 1)$.

For example:

1. 0.**0**01240191057...
2. 0.1**2**3451234512...
3. 0.33**3**333333333...
4. 0.500**0**00000000...
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7. 0.141592**6**53589...
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$\vdots$

Algorithm for producing a number that is not on the list:

In the ith number in the list, highlight the ith digit.

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i. If the highlighted digit of the ith number is a 0, then make the corresponding digit of the new number a 1.

ii. If the highlighted digit of the ith number is not a 0, then make the corresponding digit of the new number a 0.

Example:

0.1001

Take this is the supposedly complete list of real numbers in $[0, 1)$.

For example:

1. 0.**0**01240191057...
2. 0.1**2**3451234512...
3. 0.33**3**333333333...
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Algorithm for producing a number that is not on the list:

In the ith number in the list, highlight the ith digit.

Build a new number as follows:

i. If the highlighted digit of the ith number is a 0, then make the corresponding digit of the new number a 1.

ii. If the highlighted digit of the ith number is not a 0, then make the corresponding digit of the new number a 0.

Example:
0.10010

Take this is the supposedly complete list of real numbers in $[0, 1)$.

For example:

1. 0.**0**01240191057...
2. 0.1**2**3451234512...
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In this way, this new number differs from every item in the list in at least one digit!

Theorem. The set of real numbers in the interval $[0, 1)$ is not countable.

Proof outline: We will prove this by contradiction. Suppose that the set $[0, 1)$ is countable, so that the real numbers in $[0, 1)$ can be listed. Show that any list isn't complete. Namely, algorithmically produce an element of $[0, 1)$ that isn't on any fixed list.

Proof. For $x \in [0, 1)$, denote the ith digit of x by $x[i]$. Note that for $x, y \in [0, 1)$, we have $x = y$ if and only if $x[i] = y[i]$ for all $i \in \mathbb{Z}_{>0}$.

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Now, ~~suppose $f : \mathbb{Z}_{>0} \to [0, 1)$ is a bijection~~ let $f : \mathbb{Z}_{>0} \to [0, 1)$. Define $x_f \in [0, 1)$ so that the ith digit of x_f is

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For sets X and Y, we say $|X| \leqslant |Y|$ if there exists an injective function $f : X \to Y$. And write $|X| < |Y|$ if $|X| \leqslant |Y|$ and $|X| \neq |Y|$.

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Outline:

1. Show $|A| \leqslant |\mathcal{P}(A)|$ by showing an injective map *exists* (give one example).
2. Show $|A| \neq |\mathcal{P}(A)|$ by showing that *any* map (not just the example from before) cannot be surjective.

   Hint. For any $f : A \to \mathcal{P}(A)$, show f is not surjective; i.e. construct a set $S \subseteq A$ such that $S \neq f(a)$ for all $a \in A$.

(Need for care: 1 is a “there exists” statement; 2 is a “for all” statement.)

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$$(a \sim b \wedge b \sim c) \Rightarrow a \sim c$$

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An equivalence relation on a set A is a binary relation that is reflexive, symmetric, *and* transitive.

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An equivalence relation on a set A is a binary relation that is reflexive, symmetric, *and* transitive. (Only #1)

Fix $n \in \mathbb{Z}_{>0}$ and define the relation on $\mathbb{Z}$ given by

$$\text{"}a \sim b \quad \text{ if } a \equiv b \pmod{n}.\text{"}$$

Is $\sim$ is an equivalence relation?

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Is $\sim$ is an equivalence relation?

Check: we have $a \equiv b \pmod{n}$ if and only if $a - b = kn$ for some $k \in \mathbb{Z}$.

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$$\text{“}a \sim b \quad \text{ if } a \equiv b \pmod{n}.\text{”}$$

Is $\sim$ is an equivalence relation?

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Yes! This is an equivalence relation!

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So no, it is not an equivalence relation.

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Read: Why reflexivity doesn't follow from symmetry and transitivity.

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Example: Consider the equivalence relation on $A = \{a, b, c\}$ given by

$$a \sim a, \quad b \sim b, \quad c \sim c, \quad a \sim c, \quad \text{and} \quad c \sim a.$$

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are the two equivalence classes in A (with respect to this relation).

(We say there are two, *not three*, since "the equivalence classes" refers to the sets themselves, not to the elements that generate them.)

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Example: We showed that

$$\text{“}a \sim b \quad \text{if } a \equiv b \pmod{5}\text{”}$$

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We call any element a of a class C representative of C (since we can write $C = [a]$ for any $a \in C$).

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So $\boxed{\mathbb{Z} = [0] \sqcup [1] \sqcup [2] \sqcup [3] \sqcup [4]}$.