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Question: If I run a computer program 30 times (in sequence) that takes 5 hours to run each time, when will it be done?

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Recall the division algorithm: For $a, n \in \mathbb{Z}$ with $n \neq 0$, there exist unique $q, r \in \mathbb{Z}$ with $0 \leqslant r < |n|$ satisfying $a = nq + r$.

In logic:

$$\forall a \in \mathbb{Z}, n \in \mathbb{Z}_{\neq 0}, \exists! q \in \mathbb{Z}, r \in \{0, 1, \ldots, |n| - 1\} (a = bq + r).$$

($\exists!$ means "there exist(s) unique"—not only do they exist, but they're the only ones.)

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If a and b have the same remainder when divided by n, we say

a is congruent to b modulo (mod) n

written

$$a \equiv b \pmod{n} \quad \text{or} \quad a \equiv_n b.$$

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$14 \equiv 2 \pmod{12}$

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$$14 \equiv 2 \pmod{12}, \quad 26 \equiv 2 \pmod{12}$$

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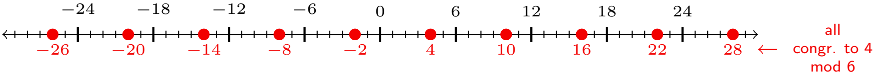
Example: The numbers that are equivalent to 4 modulo 6 are



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Lemma. For $a, b, n \in \mathbb{Z}$ with $n \neq 0$, we have $a \equiv b \pmod{n}$ if and only if $n | a - b$.

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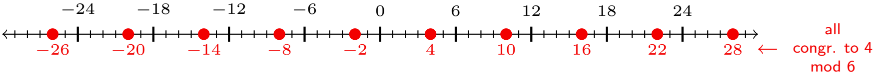

Lemma. For $a, b, n \in \mathbb{Z}$ with $n \neq 0$, we have $a \equiv b \pmod{n}$ if and only if $n | a - b$.

Proof. Fix $a, b \in \mathbb{Z}$. By the division algorithm, there exist $q_1, q_2, r_1, r_2 \in \mathbb{Z}$ with $0 \leqslant r_1, r_2 < |n|$ satisfying

$$a = q_1 n + r_1 \quad \text{and} \quad b = q_2 n + r_2.$$

If $a \equiv_n b$, then $r_1 = r_2$, so that

$$a - b = q_1 n + r_1 - (q_2 n + r_2) = (q_1 - q_2)n.$$

Since $q_1 - q_2 \in \mathbb{Z}$, we have $n | a - b$, as desired.

Conversely, if $n | a - b$, then $a - b = kn$ for some $k \in \mathbb{Z}$. Thus,

$$kn = a - b = q_1 n + r_1 - (q_2 n + r_2) = (q_1 - q_2)n + (r_1 - r_2).$$

Therefore,

$$r_1 - r_2 = (k - q_1 + q_2)n, \quad \text{so that} \quad n | r_1 - r_2.$$

But since $0 \leqslant r_1, r_2 < |n|$, we have $-|n| < r_1 - r_2 < |n|$.
Therefore, $r_1 - r_2 = 0$, so that $a \equiv b \pmod{n}$, as desired. □

For the rest of today, fix $n \in \mathbb{Z}_{\geqslant 1}$, and assume all other variables are integers.

Proposition. If $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n}$, then

(a) $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$, and

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$$a_1 - b_1 = k_1 n \quad \text{and} \quad a_2 - b_2 = k_2 n$$

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for some $k_1, k_2 \in \mathbb{Z}$. To prove the lemma, show (by direct computation) that

$$(a_1 + a_2) - (b_1 + b_2) = kn \quad \text{and } a_1 a_2 - b_1 b_2 = \ell n$$

for some $k, \ell \in \mathbb{Z}$.

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For example, $x = 2$ is a solution to $3x \equiv 6 \pmod{10}$. But so are

$$12,\ 22,\ 31,\ \ldots, \quad \text{as well as} \quad -8,\ -18,\ -28,\ \ldots.$$

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Homework: If $\gcd(c, n) = 1$, then

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Prop. If $\gcd(c, n) \neq 1$, then there are a and b such that

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Proof. Letting $\gcd(n, c) = g > 1$, there are $2 \leqslant k < n$ and $2 \leqslant \ell < c$ such that $kg = n$ and $\ell g = c$. So $ck = \ell gk = \ell n$. Therefore

$$ck \equiv_n 0 \equiv_n c \cdot 0.$$

But since $2 \leqslant k < n$, $k \not\equiv 0 \pmod{0}$.

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And if $\gcd(n, c) = 1$ and $ac \equiv bc \pmod{n}$, then $a \equiv b \pmod{n}$.

Solving congruences.

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Solving addition problems involves subtraction, which is straightforward:

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x	0	1	2	3	4	5	6	7	8	9
$4x$	0	4	8	12	16	20	24	28	32	36
least residue	0	4	8	2	6	0	4	8	2	6

If $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n}$, then

(a) $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$, and

(b) $a_1 a_2 \equiv b_1 b_2 \pmod{n}$.

And if $\gcd(n, c) = 1$ and $ac \equiv bc \pmod{n}$, then $a \equiv b \pmod{n}$.

Solving congruences.

Solving multiplication problems involves division, which is less straightforward.

Example: $4x \equiv 1 \pmod{7}$.
Since $\gcd(4, 7) = 1$, there will be a unique solution (up to congruence).
And since $1 \equiv_7 8 = 4 \cdot 2$, we have $x \equiv 2 \pmod{7}$ is that solution.

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In our example above, $5 \cdot 4 = 20 \equiv 1 \pmod{19}$.

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$$x \equiv_{19} 5 \cdot 4 \cdot x \equiv_{19} 5 \cdot 3 \equiv_{19} 15.$$

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Claim: If you want to solve congruences of the form

$$ax \equiv b \pmod{n},$$

you have two cases, based on $d = \gcd(a,n)$.

1. If $d \nmid b$, then there are no solutions.
2. If $d|b$, then there are exactly d solutions (mod n).

   To find them, compute $u, v \in \mathbb{Z}$ such that $ua + vn = d$. Then

   $$b = (b/d)d = (b/d)ua + (b/d)vn,$$

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(Very Important) Corollary. An integer b has a (unique) multiplicative inverse modulo n if and only if $\gcd(b, n) = 1$.
Namely, if p is prime, then b has a multiplicative inverse modulo p if and only if $b \not\equiv_p 0$.

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If p is prime, then $x^p \equiv_p x$ for all $x \in \mathbb{Z}$.

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We will prove the theorem for $x \geqslant 0$ by induction on x. First, we have $0^p = 0 \equiv_p 0$, as desired. Next, fix $x \geqslant 0$, and assume $x^p \equiv_p x$. Then, using the binomial theorem,

$$(x+1)^p \equiv_p \cdots \qquad \text{Homework!}$$

Wrapping up elementary number theory...

Chapters 27–29 in "How to think..." (on primes, divisors, gcd, Euclidean algorithm, modular arithmetic, etc.) are a *brief* introduction to elementary number theory.

To learn more: Take "Theory of Numbers" (Math 345).

Where else this is used: Modular arithmetic is a *Very Important Example* in "Modern Algebra" (a.k.a. "abstract algebra"), Math 347/A49.