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$$-15|60 \quad \text{since} \quad 60 = (-15) * (-4);$$

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For two numbers $a, b \in \mathbb{Z}_{>0}$, a common divisor d is a divisor common to both numbers, i.e. $d|a$ and $d|b$.

For example,

3 is a divisor of 30, but not 40;

4 is a divisor of 40, but not 30;

1, 2, 5, and 10 are all common divisors of 30 and 40.

The greatest common divisor of a and b, denoted $\gcd(a, b)$ is largest integer that divides both a and b. Ex: $\gcd(30, 40) = 10$.

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Example: The divisors of 25 are $\pm 1, \pm 5$, and ± 25; the divisors of 12 are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6$, and ± 12; so 25 and 12 are relatively prime (even though neither is prime).

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Not *computationally* efficient either, since prime factorization is computationally difficult/not possible without a list of primes.

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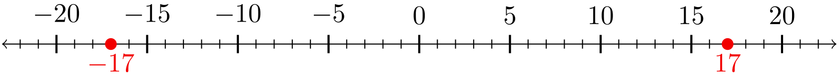

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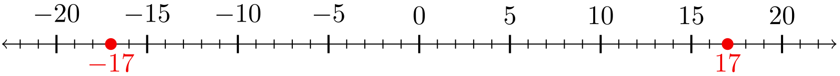

Proof: (sketch)

Case 1: If a and b are the same sign, subtract b from a until the result is between 0 and $|b| - 1$. The result is r and the number of subtractions is q.

Case 2: If they're different signs, add b to a until the result is between 0 and $|b| - 1$. The result is r and the number of additions is $-q$.

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Play this game again with new a and b:

1. Start with $a_1 = a$ and $b_1 = b$.
2. Compute the quotient q_i and remainder r_i in dividing a_i by b_i.
3. Repeat the division algorithm using $a_i = b_{i-1}$ and $b_i = r_{i-1}$.
4. Iterate until you get $r_n = 0$.
   Then compare $\gcd(a, b)$ with r_{n-1}.

For practice: Do this process with $a = 30$, $b = 12$, and then with $a = 84$, $b = 30$.

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Claim: If n is the first time that $r_n = 0$, then $r_{n-1} = \gcd(a, b)$.
Note that if $r = 0$ in the first step, then $b|n$, so $\gcd(a, b) = b$.

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Last line: $r_{n-2} = r_{n-1}q_n$.

So

$$r_{n-3} = r_{n-2}q_{n-1} + r_{n-1} = (r_{n-1}q_n)q_{n-1} + r_{n-1} = r_{n-1}(q_nq_{n-1} + 1).$$

Then

$$\begin{aligned} r_{n-4} = r_{n-3}q_{n-2} + r_{n-2} &= r_{n-1}(q_nq_{n-1} + 1)q_{n-2} + r_{n-1}q_n \\ &= r_{n-1}(q_nq_{n-1}q_{n-2} + q_{n-2} + 1). \end{aligned}$$

Why does $r_{n-1} = \gcd(a,b)$?

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And so on...

Why does $r_{n-1} = \gcd(a, b)$?

Example: We have

$$84 = 30 * 2 + 24$$
$$30 = 24 * 1 + 6$$
$$24 = 6 * 4 + 0.$$

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Why does $r_{n-1} = \gcd(a, b)$?

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$$\begin{aligned} 84 &= 30 * 2 + 24 \\ 30 &= 24 * 1 + 6 \\ 24 &= 6 * 4 + 0. \end{aligned} \qquad r_{n-1} = 6$$

So

$$30 = 24 * 1 + 6 = (6 * 4) * 1 + 6$$

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So

$$30 = 24 * 1 + 6 = (6 * 4) * 1 + 6 = 6(4 * 1 + 1) = 6 * 5$$

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Why does $r_{n-1} = \gcd(a, b)$?

Example: We have

$$\begin{aligned} 84 &= 30 * 2 + 24 \\ 30 &= 24 * 1 + 6 \\ 24 &= 6 * 4 + 0. \end{aligned} \qquad r_{n-1} = 6$$

So

$$30 = 24 * 1 + 6 = (6 * 4) * 1 + 6 = 6(4 * 1 + 1) = 6 * 5$$
$$84 = 30 * 2 + 24 = (6 * 5) * 2 + (6 * 4) = 6(5 * 2 + 4) = 6 * 24.$$

So 6 is a common divisor of 84 and 30.

For $a = 100$, $b = 36$:

$$\begin{aligned} 100 &= 36 * 2 + 28 \\ 36 &= 28 * 1 + 8 \\ 28 &= 8 * 3 + 4 \\ 8 &= 4 * 2 + 0. \end{aligned}$$

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$$100 = 36 * 2 + 28$$
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For $a = 100$, $b = 36$:

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So

$$28 = 8 * 3 + 4$$

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$$100 = 36 * 2 + 28$$
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$r_{n-1} = 4$

So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4$$

For $a = 100$, $b = 36$:

$$\begin{aligned} 100 &= 36 * 2 + 28 \\ 36 &= 28 * 1 + 8 \\ 28 &= 8 * 3 + 4 \\ 8 &= 4 * 2 + 0. \end{aligned} \qquad r_{n-1} = 4$$

So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$

For $a = 100$, $b = 36$:

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$$r_{n-1} = 4$$

So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$

$$36 = 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2)$$

For $a = 100$, $b = 36$:

$$100 = 36 * 2 + 28$$
$$36 = 28 * 1 + 8$$
$$28 = 8 * 3 + 4$$
$$8 = 4 * 2 + 0. \qquad r_{n-1} = 4$$

So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$
$$36 = 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2) = 4(7 * 1 + 2) = 4 * 9$$

For $a = 100$, $b = 36$:

$$100 = 36 * 2 + 28$$
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So

$$28 = 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7$$
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$$100 = 36 * 2 + 28$$

For $a = 100$, $b = 36$:

$$\begin{aligned} 100 &= 36 * 2 + 28 \\ 36 &= 28 * 1 + 8 \\ 28 &= 8 * 3 + 4 \\ 8 &= 4 * 2 + 0. \end{aligned} \qquad r_{n-1} = 4$$

So

$$\begin{aligned} 28 &= 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7 \\ 36 &= 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2) = 4(7 * 1 + 2) = 4 * 9 \\ 100 &= 36 * 2 + 28 = (4 * 9) * 2 + (4 * 7) \end{aligned}$$

For $a = 100$, $b = 36$:

$$100 = 36 * 2 + 28$$
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$$100 = 36 * 2 + 28 = (4 * 9) * 2 + (4 * 7) = 4(9 * 2 + 7) = 4 * 25.$$

For $a = 100$, $b = 36$:

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So

$$\begin{aligned} 28 &= 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7 \\ 36 &= 28 * 1 + 8 = (4 * 7) * 1 + (4 * 2) = 4(7 * 1 + 2) = 4 * 9 \\ 100 &= 36 * 2 + 28 = (4 * 9) * 2 + (4 * 7) = 4(9 * 2 + 7) = 4 * 25. \end{aligned}$$

So 4 is a common divisor of 100 and 36.

For $a = 100$, $b = 36$:

$$
\begin{aligned}
100 &= 36 * 2 + 28 \\
36 &= 28 * 1 + 8 \\
28 &= 8 * 3 + 4 \\
8 &= 4 * 2 + 0. \qquad\qquad r_{n-1} = 4
\end{aligned}
$$

So

$$
\begin{aligned}
28 &= 8 * 3 + 4 = (4 * 2) * 3 + 4 = 4(2 * 3 + 1) = 4 * 7 \\
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100 &= 36 * 2 + 28 = (4 * 9) * 2 + (4 * 7) = 4(9 * 2 + 7) = 4 * 25.
\end{aligned}
$$

So 4 is a common divisor of 100 and 36.

You try: use the following computations, working backwards, to show that 2 is a common divisor of 100 and 26:

$$100 = 26 * 3 + 22 \qquad 26 = 22 * 1 + 4$$
$$22 = 4 * 5 + 2 \qquad 4 = 2 * 2 + 0$$

Why does $r_{n-1} = \gcd(a, b)$?

Letting $r_{-1} = a$ and $r_0 = b$, and computing

$$\begin{array}{rcccll}
r_{-1} & = & r_0 * q_1 & + & r_1 & \\
r_0 & = & r_1 * q_2 & + & r_2 & \\
r_1 & = & r_2 * q_3 & + & r_3 & \\
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r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a, b)? \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n
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we can reverse this process to show that r_{n-1} is, at the very least, a *common divisor* to $a = r_{-1}$ and $b = r_0$.

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Wait! How do we know we ever get 0??

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we can reverse this process to show that r_{n-1} is, at the very least, a *common divisor* to $a = r_{-1}$ and $b = r_0$.

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The division algorithm ensures that each remainder is strictly smaller than the last, and always non-negative:

$$b = r_0 > r_1 > r_2 > \cdots \geqslant 0.$$

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& \vdots & & & & \\
r_{n-4} & = & r_{n-3} * q_{n-2} & + & r_{n-2} & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & \leftarrow \gcd(a, b)? \\
r_{n-2} & = & r_{n-1} * q_n & + & 0 & \leftarrow r_n
\end{array}$$

we can reverse this process to show that r_{n-1} is, at the very least, a *common divisor* to $a = r_{-1}$ and $b = r_0$.

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The division algorithm ensures that each remainder is strictly smaller than the last, and always non-negative:

$$b = r_0 > r_1 > r_2 > \cdots \geqslant 0.$$

So since the r_i's are all *integers*, this process ends at some point.

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We have that r_{n-1} is a common divisor to a an b. Now why is it the *greatest* common divisor?

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Back to our division calculation, and substitute these equations in:

$$\begin{array}{ccccc} a & = & b * q_1 & + & r_1 \\ b & = & r_1 * q_2 & + & r_2 \\ r_1 & = & r_2 * q_3 & + & r_3 \\ & \vdots & & & \\ r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} \\ r_{n-2} & = & r_{n-1} * q_n & + & 0 \end{array}$$

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$$\begin{array}{ccccc} d\alpha & = & d\beta * q_1 & + & r_1 \\ b & = & r_1 * q_2 & + & r_2 \\ r_1 & = & r_2 * q_3 & + & r_3 \\ & \vdots & & & \\ r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} \\ r_{n-2} & = & r_{n-1} * q_n & + & 0 \end{array}$$

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b & = & r_1 * q_2 & + & r_2 \\
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 & \vdots & & & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} \\
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$$\begin{array}{ccccccl}
d\alpha & = & d\beta * q_1 & + & r_1 & & \text{so } r_1 = d(\alpha - \beta q_1) = dm_1 \\
b & = & r_1 * q_2 & + & r_2 & & \\
r_1 & = & r_2 * q_3 & + & r_3 & & \\
 & \vdots & & & & & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & & \\
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r_1 & = & r_2 * q_3 & + & r_3 & & \\
 & \vdots & & & & & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & & \\
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dm_1 & = & dm_2 * q_3 & + & r_3 & & \text{so } r_3 = \cdots = dm_3 \\
 & \vdots & & & & & \\
r_{n-3} & = & r_{n-2} * q_{n-1} & + & r_{n-1} & & \\
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We have that r_{n-1} is a common divisor to a an b. Now why is it the *greatest* common divisor?

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Theorem (Euclidean algorithm). To compute $\gcd(a, b)$, let $r_{-1} = a$ and $r_0 = b$, and compute successive quotients and remainders

$$r_{i-2} = r_{i-1}q_i + r_i$$

for $i = 1, 2, 3, \ldots$, until some remainder r_n is 0. The last nonzero remainder r_{n-1} is then the greatest common divisor of a and b.

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Proof technique: The definition of greatest common divisor is an "and" statement:

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Claim (on homework): For any non-zero $a, b \in \mathbb{Z}$, there exist $x, y \in \mathbb{Z}$ such that $\gcd(a,b) = ax + by$, i.e. $\gcd(a,b)$ is an integral combination of a and b.

Theorem. For any non-zero $a, b \in \mathbb{Z}$, there exist $x, y \in \mathbb{Z}$ such that

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You try: Let a and b be non-zero integers. Outline proofs of the following claims.

1. If δ is a common divisor of a and b, then $\delta | \gcd(a, b)$.
2. We call $\ell \in \mathbb{Z}$ a common multiple of a and b if $a|\ell$ and $b|\ell$. The smallest (positive) such ℓ is called the *least common multiple* of a and b, denoted $\operatorname{lcm}(a, b)$. For example, $\operatorname{lcm}(12, 66) = 132$.
   (i) If $a|m$ and $b|m$, then $\operatorname{lcm}(a, b)|m$.
   (ii) For any $r \in \mathbb{Z}$, $\operatorname{lcm}(ra, rb) = r \operatorname{lcm}(a, b)$.