Recall proof by induction:

1. Show a base case: show that $P(n_0)$ is true.
2. Induction step: show $P(n) \Rightarrow P(n+1)$ for all $n \geqslant n_0$.

Then since $\Rightarrow$ is “transitive”, we have proven $P(k)$ for all $k \geqslant n_0$.

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For example, we proved

$$P(n): \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

by checking

$$P(1): \qquad \sum_{i=1}^{1} i = 1 = \frac{1(2)}{2} \checkmark,$$

and $P(n) \Rightarrow P(n+1)$:

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2} \checkmark.$$

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So for any $k \geqslant 1$, we have $P(1)$ is true and

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow \cdots \Rightarrow P(k-1) \Rightarrow P(k),$$

so $P(k)$ is true too!

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A variant on induction: Strong Induction

1. Show a base case: show that $P(n_0)$ is true.
2. Induction step: show

$$\left(\bigwedge_{i=n_0}^{n} P(i) \right) \Rightarrow P(n+1)$$

   for all $n \geqslant n_0$.
   "Fix $n \geqslant n_0$ and assume $P(k)$ for all $n_0 \leqslant k \leqslant n$.
   ↖*(Strong) induction hypothesis*
   Then .... So $P(n+1)$ is true." ←*Induction step*

Theorem. (The Fundamental Theorem of Arithmetic) Every integer $n \geqslant 2$ can be factored as

$$n = p_1 p_2 \cdots p_r$$

with $p_1, p_2, \ldots, p_r$ prime (not necessarily distinct).

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Proof by strong induction (1^{st} draft). Let $P(n)$ be the thm statement.

Base case: 2 is prime, so the statement holds for $n = 2$. ✓

Goal: Assume $P(k)$ for all $2 \leqslant k \leqslant n$, and show $n + 1$ has a factorization into primes.

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Base case: 2 is prime, so the statement holds for $n = 2$. ✓
Goal: Assume $P(k)$ for all $2 \leqslant k \leqslant n$, and show $n+1$ has a factorization into primes.
Induction step: Fix $n \geqslant 2$ and assume that k has a prime factorization for all $2 \leqslant k \leqslant n$. Now consider $n+1$. Either $n+1$ is prime, or $n+1 = xy$ for some integers $2 \leqslant x, y \leqslant n$. By the induction hypothesis, there are primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$ so that

$$x = p_1 p_2 \cdots p_r \quad \text{and} \quad y = q_1 q_2 \cdots q_s.$$

So

$$n + 1 = xy = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s. \checkmark$$

Conclusion. Since $P(2)$ holds, and $\left(\bigwedge_{k=n_0}^{n} P(i)\right) \Rightarrow P(n+1)$ for all $n \geqslant 2$, we have $P(\ell)$ for all $\ell \geqslant 2$.

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Proof by strong induction (1^{st} draft).

Basically, we just showed

$$\begin{aligned}
&P(2) \\
P(2) &\Rightarrow P(3) \\
\big(P(2) \wedge P(3)\big) &\Rightarrow P(4) \\
\big(P(2) \wedge P(3) \wedge P(4)\big) &\Rightarrow P(5) \\
\big(P(2) \wedge P(3) \wedge P(4) \wedge P(5)\big) &\Rightarrow P(6) \\
&\vdots \\
\big(P(2) \wedge P(3) \wedge \cdots \wedge P(n)\big) &\Rightarrow P(n+1)
\end{aligned}$$

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Proof by strong induction (Final draft). First, 2 is prime, so the statement holds for $n = 2$. Next, fix $n \geqslant 2$ and assume k has a prime factorization for all $2 \leqslant k \leqslant n$. Now, considering $n + 1$, either $n + 1$ is prime, or $n + 1 = xy$ for some integers $2 \leqslant x, y \leqslant n$. By the induction hypothesis, there are primes $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$ so that

$$x = p_1 p_2 \cdots p_r \quad \text{and} \quad y = q_1 q_2 \cdots q_s.$$

So

$$n + 1 = xy = p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s. \checkmark$$

Thus, by strong induction, the claim holds for all $n \geqslant 2$.

Claim: A chocolate bar consists of unit squares arranged in an $n \times m$ rectangular grid. There's a way to split the bar into individual unit squares by breaking along the lines, in exactly $mn - 1$ breaks.

Proof. Let $P(m, n)$ be the statement that an $n \times m$ bar can be broken into 1×1 pieces in $mn - 1$ breaks.

Base case: If $m = n = 1$, then no breaks are required, and $1 \cdot 1 - 1 = 0$.

Inductive step: Fix $m, n \geqslant 1$, and assume that $P(k, \ell)$ holds for all $1 \leqslant k \leqslant m$ and $1 \leqslant \ell \leqslant n$. We will show $P(m + 1, n)$ and $P(m, n + 1)$ individually.

To show $P(m + 1, n)$, first make a break along a column into two pieces—am $m_1 \times n$ and an $m_2 \times n$ piece, with $1 \leqslant m_1, m_2 \geqslant m$ and $m_1 + m_2 = m + 1$. So, by the (strong) induction hypothesis, $P(m_1, n)$ and $P(m_2, n)$ both hold. So we can break the two pieces in $m_1n - 1$ and $m_2n - 1$ breaks, respectively, totaling

$$m_1n - 1 + m_2n - 1 + 1 = (m_1 + m_2)n - 1 = (m + 1)n - 1$$

moves in total. Showing $P(m, n + 1)$ follows similarly.

So by strong induction, $P(m, n)$ holds for all $m, n \geqslant 1$.

You try: Use strong induction to show the following claims.

1. Every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.
2. In the game Nim, there are two players and two piles of matches. At each turn, a player removes some (non-zero) number of matches from one of the piles. The player who removes the last match wins.
   Claim: If the two piles contain the same number of matches at the start of the game, then the second player can always win.

Introduction to number theory

An integer a *divides* the integer b, written as $a|b$, if there exists an integer k such that $b = ka$:

$$(a|b) \Leftrightarrow \exists k \in \mathbb{Z}(b = ka).$$

If $a|b$, we call a a *divisor* of b.
If a is not a divisor of b, we write $a \nmid b$ (LaTeX: \nmid).

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4. The divisors of 1 are 1 and -1.

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5. Every integer $k \in \mathbb{Z}$ is a divisor of 0 since $k \cdot 0 = 0$.
6. Zero is only a divisor of itself, i.e. $0 \nmid k$ for all $k \in \mathbb{Z}_{\neq 0}$.

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You try: Let $a, b, c \in \mathbb{Z}$. Prove the following two claims.

Claim 1: If a divides b, then a divides b^2.

Claim 2: If $a|b$ and $b|c$, then $a|c$.

Hint: for each, if $x|y$, the first thing you want to try is writing "there exists $z \in \mathbb{Z}$ such that $xz = y$."

The greatest common divisor of two non-zero integers a and b, denoted $\gcd(a, b)$, is the largest positive integer that divides both numbers:

$$\Big(\gcd(a, b) = D\Big) \Leftrightarrow \Big((d|a \wedge d|b) \Rightarrow d \leqslant D\Big).$$

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Note, for all non-zero integers a and b, we have $\gcd(a, b) > 0$.

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Note, for all non-zero integers a and b, we have $\gcd(a,b) > 0$.

If $\gcd(a,b) = 1$, we say a and b are relatively prime.