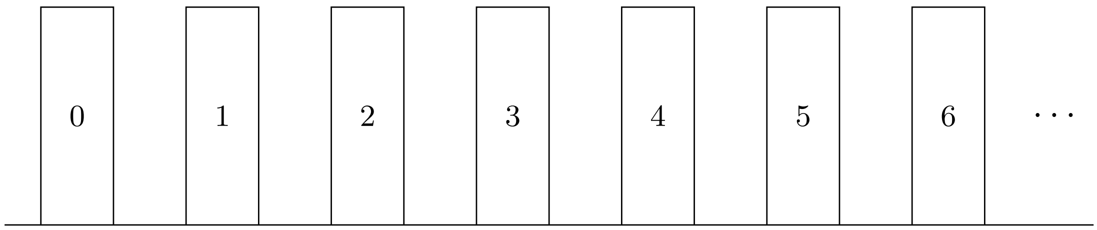
Mathematical induction

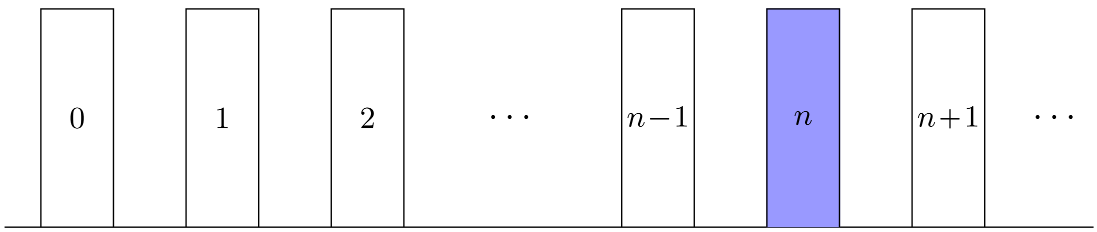
Say we have a statement, $P(n)$, that has the natural numbers $n \in \mathbb{Z}_{\geqslant 0}$ as an input.

For example, say you have an infinite row of dominoes, labeled $0, 1, 2, \ldots$:



Let $P(n)$ be the statement

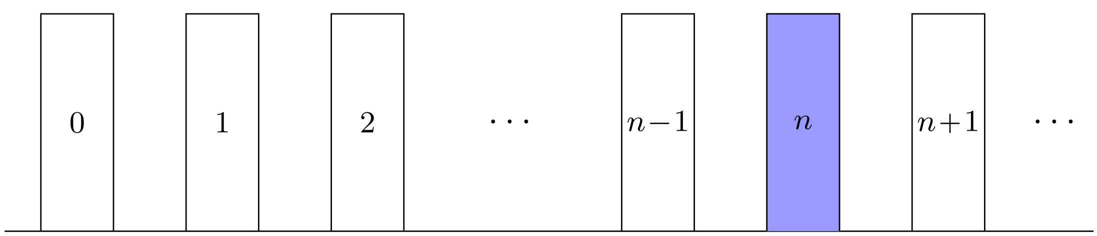
"I can knock the nth domino over".



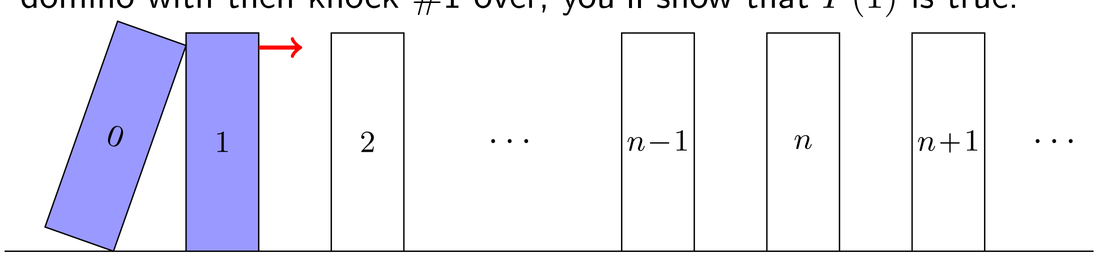
Mathematical induction

Let $P(n)$ be the statement

"I can knock the nth domino over".



Then, if you can show that the 0th domino knocking into the 1st domino with then knock #1 over, you'll show that $P(1)$ is true:



In math: You can show that $P(1)$ is true by proving
(a) $P(0)$ is true, and (b) that $P(0)$ implies $P(1)$.

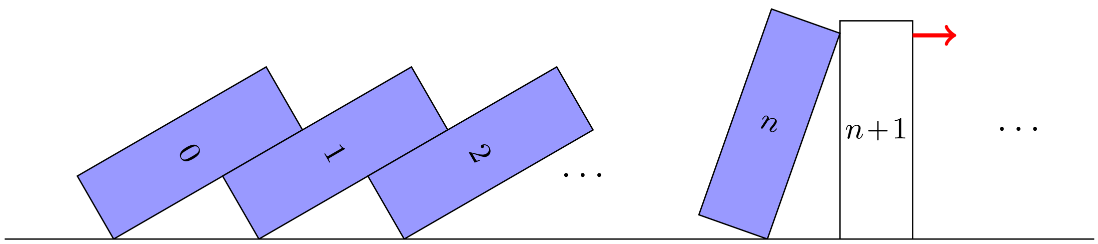
Idea: $P(1)$ will imply $P(2)$, which will imply $P(3)$, and so on...

Mathematical induction

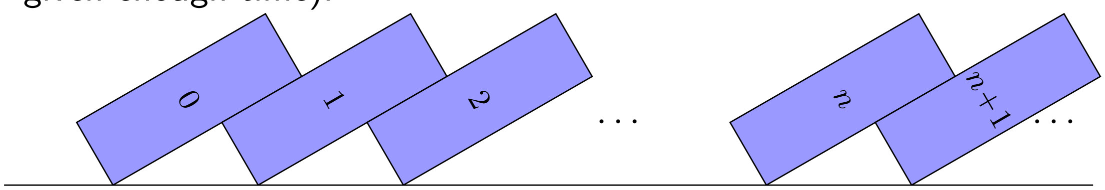
To show that $P(k)$ holds in general, you show that

(a) $P(0)$ is true, and then

(b) for any n, if $P(n)$ is true, then that implies $P(n+1)$ is also true. (If the nth domino falls, then so will the $(n+1)$th)



Then by letting the dominos fall one after the other, eventually each domino will fall (no particular domino will be left standing, given enough time):



Mathematical induction

Theorem: for any $k \in \mathbb{Z}_{\geqslant 0}$, I can knock down the kth domino.

Poof by induction:

First, I can knock down the 0th domino. ("Base case")

Now, for some $n \in \mathbb{Z}_{\geqslant 0}$, suppose I can knock down the nth domino.

("Induction hypothesis")

The nth domino will bump into the $(n+1)$th domino, which will knock it over. So that implies I can knock down the $(n+1)$th domino. ("Induction step")

Thus, by induction, I can knock down the kth domino for any $k \in \mathbb{Z}_{\geqslant 0}$. $\square$ ("Conclusion")

Math example: Show $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction.

Proof by induction (first draft).

Define $P(n)$: $P(n)$ is "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$".

Base case: The lowest case is $P(1)$, so we check that:

$$\sum_{i=1}^{1} i = \frac{1 * 2}{2}. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad \sum_{i=1}^{n+1} i = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

Inductive step: (Assume $P(n)$ and show $P(n+1)$)
Fix $n \geqslant 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (this is the Inductive Hypothesis, IH).

Math example: Show $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction.
Proof by induction (first draft). (Continued from previous slide, where $P(n)$ is "$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$"...)

Inductive step: (Assume $P(n)$ and show $P(n+1)$)
Fix $n \geqslant 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (this is the Inductive Hypothesis, IH). Then

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \underbrace{1 + 2 + \cdots + n}_{\sum_{i=1}^{n} i} + (n+1) \\ &\overset{\text{IH}}{=} \frac{n(n+1)}{2} + (n+1) \quad \text{(by the Inductive Hypothesis)} \\ &= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}. \qquad \checkmark\end{aligned}$$

Conclusion: So since $P(1)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k = 1, 2, 3, \ldots$. $\square$

Math example: Show $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ by induction.

Proof by induction (final draft). For $n = 1$, we have

$$\sum_{i=1}^{1} i = 1 = \frac{1 * 2}{2},$$

as desired. Now fix $n \geqslant 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (for that value of n). Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i &= \underbrace{1 + 2 + \cdots + n}_{\sum_{i=1}^{n} i} + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the inductive hypothesis)} \\
&= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}.
\end{aligned}$$

Thus, the claim holds for all $n \geqslant 1$ by induction. □

Example: Show $n < 2^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (first draft).

Define $P(n)$: $P(n)$ is "$n < 2^n$".

Base case: The least value of n is 0, so the base case is $P(0)$:

$$0 < 1 = 2^0. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad n+1 < 2^{n+1}.$$

Inductive step: (Assume $P(n)$ and show $P(n+1)$)
Fix $n \geqslant 0$ and assume $n < 2^n$ (this is the IH). Then since $n \geqslant 0$,

$$n+1 \overset{\text{IH}}{<} 2^n + 1 \leqslant 2^n + 2^n = 2(2^n) = 2^{n+1}. \qquad \checkmark$$

Conclusion: So since $P(0)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geqslant 0}$. □

Example: Show $n < 2^n$ for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (final draft).
For $n = 0$, we have

$$0 < 1 = 2^0,$$

as desired. Now, fix $n \geqslant 0$ and assume $n < 2^n$ (for that n). Then since $n \geqslant 0$, we have

$$n+1 < 2^n + 1 \leqslant 2^n + 2^n = 2(2^n) = 2^{n+1}.$$

Thus, the claim holds for all $n \geqslant 0$ by induction. □

Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (first draft).

Define $P(n)$: $P(n)$ is "$n^2 + n = 2k$ for some integer k".

Base case: (Show $P(0)$) We have

$$0^2 + 0 = 0 = 2 * 0. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad (n+1)^2 + (n+1) = 2\ell \text{ for some } \ell \in \mathbb{Z}$$

(Careful!! Don't use the same letter for the IH and $P(n+1)$ since it's *any* integer, not something we get from a formula!!)

Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (first draft). (Continued from previous slide, where $P(n)$ is "$n^2 + n = 2k$ for some integer k".)

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad (n+1)^2 + (n+1) = 2\ell \text{ for some } \ell \in \mathbb{Z}$$

Inductive step: (Assume $P(n)$ and show $P(n+1)$)
Fix $n \geqslant 0$ and assume $n^2 + n = 2k$ for some $k \in \mathbb{Z}$ (this is the IH). Then

$$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = \underbrace{(n^2 + n)}_{\text{even by IH}} + (2n+2)$$

$$\overset{\text{IH}}{=} 2k + 2(n+1) = 2\underbrace{(k + n + 1)}_{\in \mathbb{Z}}. \qquad \checkmark$$

Conclusion: So since $P(0)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geqslant 0}$. □

Example: Show $n^2 + n$ is even for all $n \in \mathbb{Z}_{\geqslant 0}$ by induction.

Proof by induction (final draft). For $n = 0$, we have

$$0^2 + 0 = 0 = 2 * 0,$$

as desired. Next, fix $n \geqslant 0$ and assume $n^2 + n$ is even. Then $n^2 + n = 2k$ for some $k \in \mathbb{Z}$, so that

$$\begin{aligned} (n+1)^2 + (n+1) &= n^2 + 2n + 1 + n + 1 = (n^2 + n) + (2n + 2) \\ &= 2k + 2(n+1) \qquad \text{by the inductive hypothesis,} \\ &= 2(k + n + 1). \end{aligned}$$

So since $k + n + 1 \in \mathbb{Z}$, we have $(n+1)^2 + (n+1)$ is even as well. Thus, the claim holds for all $n \geqslant 0$ by induction. □

Of course, we could have shown this directly!

Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Proof by induction (first draft).

Define $P(n)$: $P(n)$ is "if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$".

Base case: The smallest set is the empty set, so the base case is $P(0)$. In fact, the only set of size 0 is $\varnothing$. So we check $P(0)$ by computing $|\mathcal{P}(\varnothing)|$:

$$|\mathcal{P}(\varnothing)| = |\{\varnothing\}| = 1 = 2^0. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad \text{if } |B| = n+1, \text{ then } |\mathcal{P}(B)| = 2^{n+1}.$$

(Careful!! Don't use the same set name for the IH and $P(n+1)$ since they must be different sets!!)

Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Proof by induction (first draft). (Continued from previous slide, where $P(n)$ is "if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$")

Inductive step: (Assume $P(n)$ and show $P(n+1)$)

For any set A of size n, assume $|\mathcal{P}(A)| = 2^n$. Now let B be a set of size $n+1$, and let $b \in B$. Let $A = B - \{b\}$, so that

$$|A| = n \quad \text{and} \quad B = A \cup \{b\}.$$

Then for each subset $X \subseteq A$, there are exactly two subsets of B:

$$X \qquad \text{and} \qquad X \cup \{b\}.$$

So

$$|\mathcal{P}(B)| = 2|\mathcal{P}(A)| \overset{\text{IH}}{=} 2 * 2^n = 2^{n+1}. \quad \checkmark$$

Conclusion: So since $P(0)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geqslant 0}$. $\square$

Example: Show that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$.

Proof by induction (final draft). For $n = 0$, we have $A = \varnothing$, and so $\mathcal{P}(A) = \{\varnothing\}$. Thus

$$|\mathcal{P}(\varnothing)| = |\{\varnothing\}| = 1 = 2^0,$$

as desired. Now fix $n \geqslant 0$ and assume for any size-n set A, we have $|\mathcal{P}(A)| = 2^n$. Let B be a set of size $n+1$, and let $b \in B$. Let $A = B - \{b\}$, so that

$$|A| = n \quad \text{and} \quad B = A \cup \{b\}.$$

Then for each subset $X \subseteq A$, there are exactly two subsets of B:

$$X \qquad \text{and} \qquad X \cup \{b\}.$$

So

$$|\mathcal{P}(B)| = 2|\mathcal{P}(A)| = 2 * 2^n = 2^{n+1},$$

by the induction hypothesis. Thus the claim holds for all $n \geqslant 0$ by induction. □

Proof by induction

Outlining your proof:

1. Define $P(n)$.
2. Compute base case.
3. Explicitly state your goal.
4. Do inductive step.
5. State your conclusion.

Rewrite your proof:

1. Write the base case.
2. Fix n and make your inductive hypothesis.
3. Show that the claim holds for $n+1$.
4. State your conclusion.

You try:

Outline a proof by induction for the following claims.

(a) For $n \in \mathbb{Z}_{\geqslant 0}$ and $r \neq 1$, we have

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

(b) We have $n^3 + 2n$ is a multiple of 3 for all $n \in \mathbb{Z}_{\geqslant 0}$.

(c) We have $\sum_{i=1}^{n} 2i - 1 = n^2$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

(d) Suppose $A_1, A_2, \ldots A_N$ and $B_1, B_2, \ldots, B_N$ are sets such that

$$A_i \subseteq B_i \qquad \text{for all } 1 \leqslant i \leqslant N.$$

Then

$$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i.$$

(e) Suppose $A_1, A_2, \ldots A_N$ and B are sets. Then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_N - B)$$
$$= (A_1 \cap A_2 \cap \cdots \cap A_N) - B.$$