Mathematical Induction

Sorites paradox: If $1,000,000$ grains of sand forms a "heap of sand", and removing one grain from a heap leaves it a heap, then a single grain of sand (or even no grains) still forms a heap.

Mathematical induction

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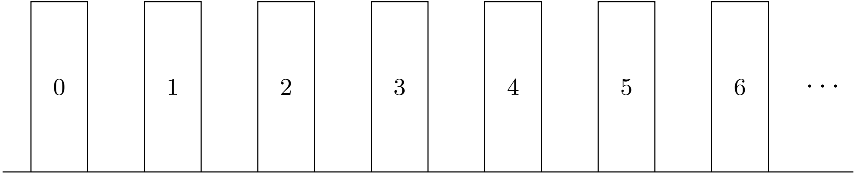

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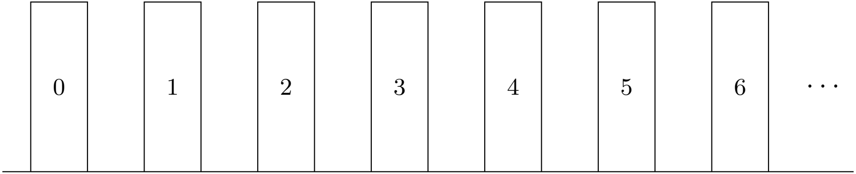
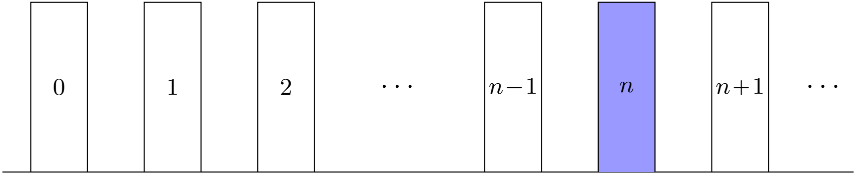


Let $P(n)$ be the statement

"I can knock the nth domino over".



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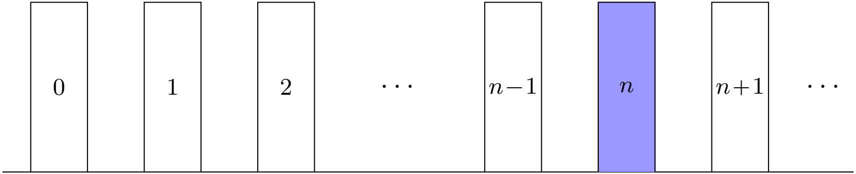
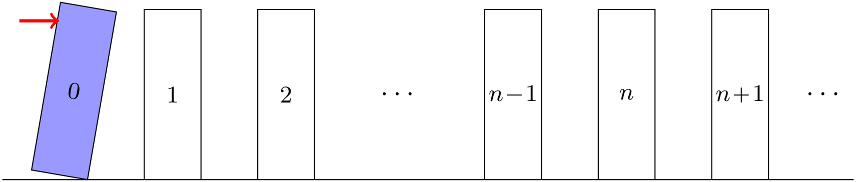


If you can start by bumping the 0th domino over, that's showing that $P(0)$ is true:



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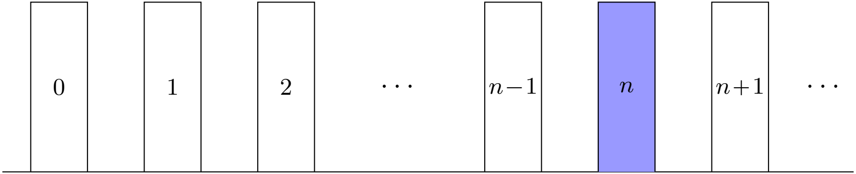
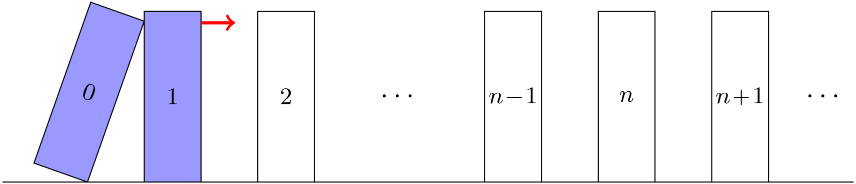


Then, if you can show that the 0th domino knocking into the 1st domino with then knock #1 over, you'll show that $P(1)$ is true:



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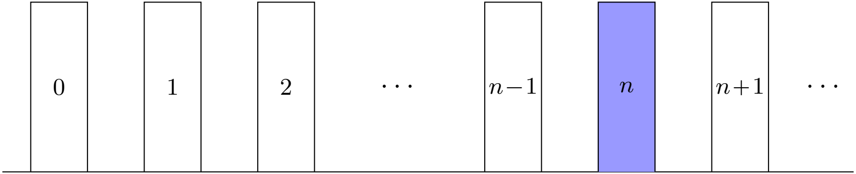

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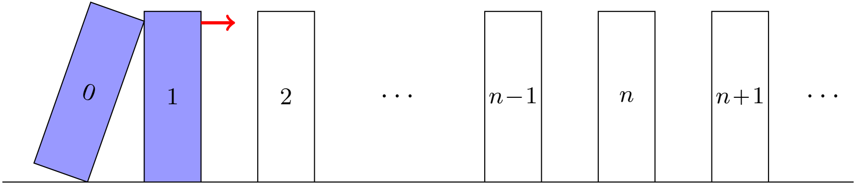

In math: You can show that $P(1)$ is true by proving (a) $P(0)$ is true, and (b) that $P(0)$ implies $P(1)$.

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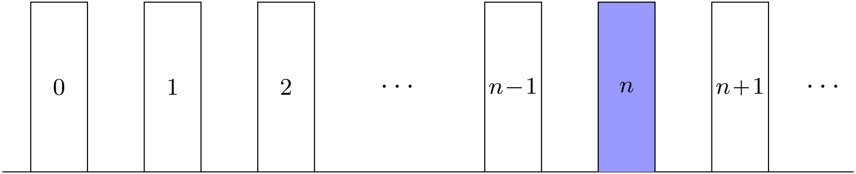

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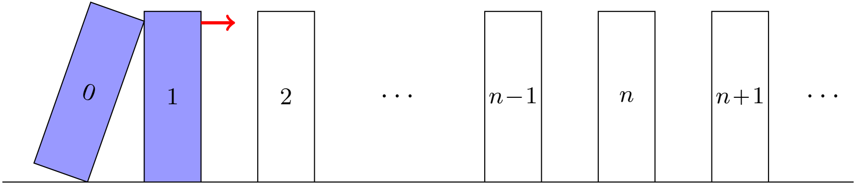

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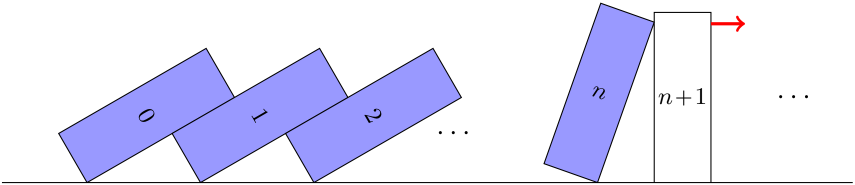
Idea: $P(1)$ will imply $P(2)$, which will imply $P(3)$, and so on...

Mathematical induction

To show that $P(k)$ holds in general, you show that

(a) $P(0)$ is true, and then

(b) for any n, if $P(n)$ is true, then that implies $P(n+1)$ is also true. (If the nth domino falls, then so will the $(n+1)$th)



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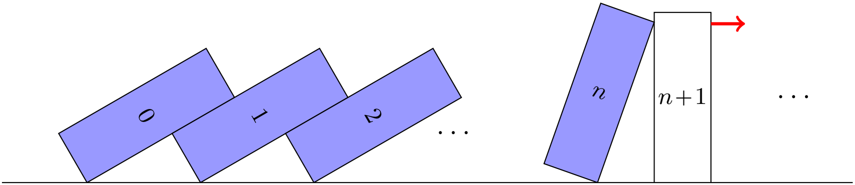
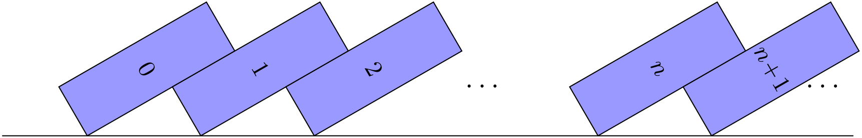


Then by letting the dominos fall one after the other, eventually each domino will fall (no particular domino will be left standing, given enough time):



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Thus, by induction, I can knock down the kth domino for any $k \in \mathbb{Z}_{\geqslant 0}$. $\square$

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Conclusion: So since $P(1)$ is true, and $P(n)$ implies $P(n+1)$, we have $P(k)$ is true for all $k = 1, 2, 3, \ldots$. $\square$

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Proof by induction (final draft). For $n = 1$, we have

$$\sum_{i=1}^{1} i = 1 = \frac{1 * 2}{2},$$

as desired. Now fix $n \geqslant 1$ and assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (for that value of n). Then

$$\begin{aligned} \sum_{i=1}^{n+1} i &= \underbrace{1 + 2 + \cdots + n}_{\sum_{i=1}^{n} i} + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \quad \text{(by the inductive hypothesis)} \\ &= \frac{n^2 + n + 2n + 2}{2} = \frac{n^2 + 3n + 2}{2} = \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Thus, the claim holds for all $n \geqslant 1$ by induction. $\square$

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Of course, we could have shown this directly!

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Proof by induction (final draft). For $n = 0$, we have $A = \varnothing$, and so $\mathcal{P}(A) = \{\varnothing\}$. Thus

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as desired. Now fix $n \geqslant 0$ and assume for any size-n set A, we have $|\mathcal{P}(A)| = 2^n$. Let B be a set of size $n+1$, and let $b \in B$. Let $A = B - \{b\}$, so that

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Outlining your proof:

1. Define $P(n)$.
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Rewrite your proof:

1. Write the base case.
2. Fix n and make your inductive hypothesis.
3. Show that the claim holds for $n + 1$.
4. State your conclusion.

You try:

Outline a proof by induction for the following claims.

(a) For $n \in \mathbb{Z}_{\geqslant 0}$ and $r \neq 1$, we have

$$\sum_{i=0}^{n} r^i = \frac{r^{n+1} - 1}{r - 1}.$$

(b) We have $n^3 + 2n$ is a multiple of 3 for all $n \in \mathbb{Z}_{\geqslant 0}$.

(c) We have $\sum_{i=1}^{n} 2i - 1 = n^2$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

(d) Suppose $A_1, A_2, \ldots A_N$ and $B_1, B_2, \ldots, B_N$ are sets such that

$$A_i \subseteq B_i \qquad \text{for all } 1 \leqslant i \leqslant N.$$

Then

$$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i.$$

(e) Suppose $A_1, A_2, \ldots A_N$ and B are sets. Then

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_N - B) = (A_1 \cap A_2 \cap \cdots \cap A_N) - B.$$