Proof by cases

Claim: For all integers n, $n^2 + 3n + 7$ is odd.

Proof (by cases). If $n \in \mathbb{Z}$, then either n is even or n is odd.

If n is even, then

$$n = 2k \qquad \text{for some } k \in \mathbb{Z}.$$

Thus

$$n^2 + 3n + 7 = (2k)^2 + 3(2k) + 7 = 2(2k^2 + 3k + 3) + 1.$$

So since $2k^2 + 3k + 3 \in \mathbb{Z}$, we have $n^2 + 3n + 7$ is odd.

Similarly, if n is odd, then

$$n = 2k + 1 \qquad \text{for some } k \in \mathbb{Z}.$$

Thus

$$n^2 + 3n + 7 = (2k+1)^2 + 3(2k+1) + 7 = 2(2k^2 + 5k + 5) + 1.$$

So since $2k^2 + 5k + 5) \in \mathbb{Z}$, we have $n^2 + 3n + 7$ is odd. □

Proof by cases

Claim: For all integers n, $n^2 + 3n + 7$ is odd.

Proof (direct). First, we note that

$$n^2 + 3n + 7 = n(n+3) + 7.$$

We have seen if a is even and b is odd, then $a + b$ is odd. So one of n or $n + 3$ is even and the other is odd. Thus $n(n+3)$ is even; and hence $n(n+3) + 7$ is odd. □

Note: we kind of cheated by using previously formed machinery! We used the lemma,

"If a is even and b is odd, then $a + b$ is odd."

Our first proof was called a proof by cases or proof by exhaustion. It was a little more obvious, but not as illuminating.

Tip: Anything with multiples or remainders lend themselves to proof by cases.

Claim: For all $n \in \mathbb{Z}$, $n^3 - n$ is a multiple of 3.

Proof (by cases). Every $n \in \mathbb{Z}$ can be written as $3n + r$ for $r = 0, 1$, or 2 (r is the remainder of n divided by 3).

Case 1: $r = 0$. Then...
Case 2: $r = 1$. Then...
Case 3: $r = 2$. Then...

(See Example 22.3 for details.)

Proof (by cases, but more direct). Note that

$$n^3 - n = n(n + 1)(n - 1).$$

Then we will have our desired result by showing that one of n, $n + 1$, or $n - 1$ must be a multiple of three. Namely, every $n \in \mathbb{Z}$ can be written as $3n + r$ for $r = 0$, 1, or 2. But if n and $n + 1$ are not multiples of 3, then r must be 1, so that $n - 1$ is a multiple of 3. □

Note: Again, we cheated, this time by assuming our reader was knowledgeable enough to understand second and fourth sentences.

Tip: Piecewise functions lend themselves to proof by cases.

Theorem (Triangle inequality). For real numbers x and y, we have

$$|x + y| \leqslant |x| + |y|.$$

Note: absolute value is secretly a piecewise function!

$$|x| = \begin{cases} x & x \geqslant 0 \\ -x & x < 0. \end{cases}$$

Proof (by cases). Suppose, without loss of generality, that $x \geqslant y$.
Case 1: $x, y \geqslant 0$. If $x, y \geqslant 0$, then $x + y \geqslant 0$. So

$$|x| = x \quad |y| = y \quad \text{and} \quad |x + y| = x + y = |x| + |y|.$$

Case 2: $y < 0$ and $x \geqslant 0$. (See Theorem 22.7)

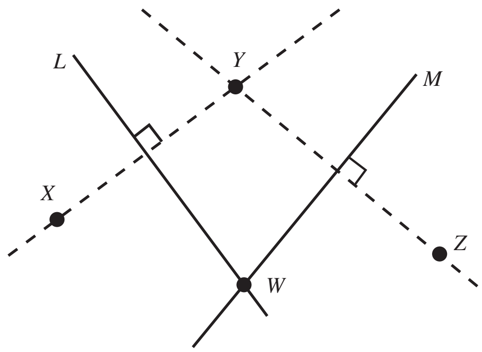
Case 3: $x, y < 0$. (See Theorem 22.7)

Note: The triangle inequality is an important part of the definition of a metric, i.e. a distance function. Absolute value is just one metric on $\mathbb{R}$.

Theorem. Let X, Y and Z be points in the plane. Then, all lie on a line or all lie on a circle.

Proof. Denote by L the perpendicular bisector between X and Y, and by M the perpendicular bisector between Y and Z. If the points all lie on a line, then they satisfy the conclusion of the theorem and we are done. If the points are not on a line, then L and M are not parallel and so must meet at a point W, say.

As W is on the perpendicular bisector L it is equidistant from X and Y. Similarly it is equidistant from Y and Z. As the distance from W to X and the distance from Y and Z are the same we deduce that the three points lie on a circle centered at W of radius equal to the distance from W to X. □



Theorem. Let X, Y and Z be points in the plane. Then, all lie on a line or all lie on a circle.

Tip: "Or" statements lend themselves to proof by cases.

Tip: Extreme examples lend themselves to proof by cases.

You try:

Outline a proof by cases for each of the following statements.

1. The square of any integer is of the form $3k$ or $3k+1$ for some $k \in \mathbb{Z}$.
2. The cube of any integer is of the form $9k$, $9k+1$, or $9k+8$ for some $k \in \mathbb{Z}$.
3. For all $x, y \in \mathbb{R}$, we have
$$|xy| = |x||y| \qquad \text{and} \qquad \big||x| - |y|\big| \leqslant |x - y|.$$
4. For sets A and B, we have $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$.

Tip: Unions lend themselves to proof by cases (since they're secretly "or" statements).