Worksheet

Proof (of Pythagoras' Theorem). The proof can be shown using the two squares in Figure 19.3. To draw the first square begin by drawing a general triangle with sides a and b and then extend these edges by lengths b and a respectively. Then we can complete the drawing to get the square on the left-hand side of Figure 19.3.

We can draw another square like the one on the right-hand side of the figure. From the figure we can see that both squares have equal area and so we can conclude that

$$\begin{aligned}\text{Area of left square} &= \text{Area of right square}\\ c^2 + (4 \times \text{Area of } (a, b)\text{-triangle}) &= a^2 + b^2 + (4 \times \text{Area of } (a, b)\text{-triangle})\\ c^2 &= a^2 + b^2.\end{aligned}$$

□

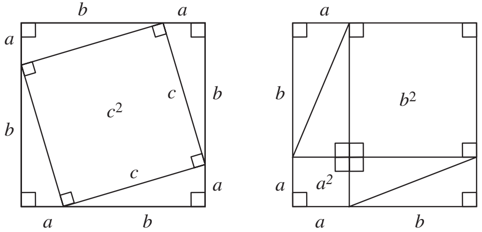

Figure 19.3 Proof of Pythagoras' Theorem

Proof by cases

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Our first proof was called a proof by cases or proof by exhaustion. It was a little more obvious, but not as illuminating.

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Note: Again, we cheated, this time by assuming our reader was knowledgeable enough to understand second and fourth sentences.

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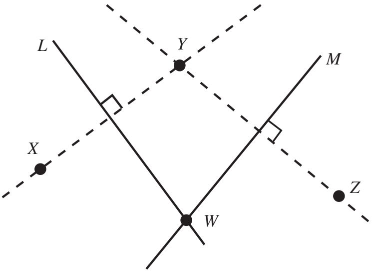
Proof. Denote by L the perpendicular bisector between X and Y, and by M the perpendicular bisector between Y and Z. If the points all lie on a line, then they satisfy the conclusion of the theorem and we are done. If the points are not on a line, then L and M are not parallel and so must meet at a point W, say.

As W is on the perpendicular bisector L it is equidistant from X and Y. Similarly it is equidistant from Y and Z. As the distance from W to X and the distance from Y and Z are the same we deduce that the three points lie on a circle centered at W of radius equal to the distance from W to X. □



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Tip: "Or" statements lend themselves to proof by cases.

Tip: Extreme examples lend themselves to proof by cases.

You try:

Outline a proof by cases for each of the following statements.

1. The square of any integer is of the form $3k$ or $3k+1$ for some $k \in \mathbb{Z}$.
2. The cube of any integer is of the form $9k$, $9k+1$, or $9k+8$ for some $k \in \mathbb{Z}$.
3. For all $x, y \in \mathbb{R}$, we have
$$|xy| = |x||y| \qquad \text{and} \qquad \big||x| - |y|\big| \leqslant |x - y|.$$
4. For sets A and B, we have
$A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$.

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Tip: Unions lend themselves to proof by cases (since they're secretly "or" statements).