Definitions, theorems, and proofs

- Definition: an explanation of the mathematical meaning of a word.
- Theorem: a very important true statement.
- Proposition: a less important but nonetheless interesting true statement.
- Lemma: a true statement used in proving other true statements.
- Corollary: a true statement that is a simple deduction from a theorem or proposition.
- Proof: the explanation of why a statement is true.
- Conjecture: a statement believed to be true, but for which we have no proof.
- Axiom: a basic assumption about a mathematical situation.

How to read a theorem

1. Find the assumptions and conclusions
2. Rewrite in symbols or in words
3. Observe the detail
4. What does the theorem do and how it can be used?
5. Draw a picture
6. Apply to trivial examples and other extreme cases
7. Compare with earlier theorems
8. Rate the "strength" of the assumptions and conclusions
    - Is the converse true?
    - What happens to non-examples?
    - Can you generalize

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Theorem 1: Let $f : X \to Y$ be a function between sets X and Y, and let $A, B \subseteq X$. Then $f(A \cup B) = f(A) \cup f(B)$.
Theorem 2: Let x be a natural number. If x is odd, then x^3 is odd.
Theorem 3: The square root of 2 is irrational.
Theorem 4: We have $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Proofs

"Do not confuse reasons which sound good with good, sound reasons."

-Anon.

What *is* a proof?

A *proof* is an explanation of why a statement is true.

It must be **convincing and rigorous**.

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What *isn't* a proof?

It is not enough to just provide evidence (examples); it is also not enough to display logical steps that aren't readable. A good proof must tell a story; but also be irrefutable.

Also, a proof is *not* a tutorial in problem-solving. "Solving the problem" is the precursor to writing the proof.

Read Chapter 17.

How to read a proof

Review Chapter 2!

1. Skim through and identify what is important.
2. Ask questions.
3. Read through more carefully.
4. Be active. This should include checking the text and doing the exercises.
5. Reflect.

Thm. Let m and n be natural numbers. The product mn is odd if and only if m and n are odd.

Proof. Suppose that m and n are odd. Then, by definition, $m = 2k + 1$ and $n = 2j + 1$ for some natural numbers k and j. Then $mn = 2(2kj + j + k) + 1$. Since $2kj + j + k$ is a natural number, we have mn is odd.

Now assume that one of m and n is even. Without loss of generality, we can assume m is even, so $m = 2k$ for some natural number k. We have $mn = 2kn$. So since kn is a natural number, mn is even. Hence, the only way mn can be odd is if m and n are odd.

Therefore, mn is odd if and only if m and n are odd. $\square$

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Skimming.

1. Break into pieces/steps.
2. Identify the methods used.
3. Find where the assumptions are used.

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Asking questions/being active.

1. Do simple examples.
2. Draw pictures.
3. Apply to a non-example.

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Reading more carefully/being active.

1. Check the text.
2. Look for mistakes.
3. Try extreme examples.

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Reflecting.

1. Is the proof like any other proof you have seen?
2. Can this proof be strengthened? Generalized?

Worksheet