Examples: what, why, and when

Examples are important for...

- processing and learning new ideas;
- testing statements, i.e. testing that we are thinking logically;
- remembering definitions and statements;
- explaining statements, proofs, etc.

Even the process of trying to *find* examples or counterexamples can help you figure out a proof.

Doing examples before proving a claim can

- convince you that it's true;
- help you spot potential pitfalls in your proof;
- help you decide whether you need to tackle a problem in separate cases or not.

Doing examples usually cannot substitute a proof.

Collect favorite examples

When building examples of sets, consider

- finite sets, infinite sets;
- equal, disjoint, subsets, generic.

When building examples of functions, consider

- injective, surjective, both, neither;
- trivial, identity;
- formulas, pictures.

Counterexamples

An example that shows that a statement is false is called a counterexample.

- Consider ‘All primes are odd.’ The number 2 is a counterexample to this statement.
- Consider ‘Let p and q be real numbers. If $p/q \in \mathbb{Q}$, then $p \in \mathbb{Q}$ and $q \in \mathbb{Q}$.’ The numbers $p = \pi/3$ and $q = \pi/2$ are a counterexample to this statement.
- Consider ‘If $x^2 = x$ then $x = 1$’. The value $x = 0$ is a counterexample to this statement.

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You try: Restate each of the following as ‘If ..., then ...’, and give a counterexample.

1. The square root of every integer is an integer.
2. $x^3 \leqslant 0$ for all $x \leqslant 1$.
3. For all $x, y \in \mathbb{R}_{\geqslant 0}$ we have $\sqrt{x^2 + y^2} = x + y$.

Definitions, theorems, and proofs

- Definition: an explanation of the mathematical meaning of a word.
- Theorem: a very important true statement.
- Proposition: a less important but nonetheless interesting true statement.
- Lemma: a true statement used in proving other true statements.
- Corollary: a true statement that is a simple deduction from a theorem or proposition.
- Proof: the explanation of why a statement is true.
- Conjecture: a statement believed to be true, but for which we have no proof.
- Axiom: a basic assumption about a mathematical situation.

Propositions versus Theorems:
As the book says, there isn't a precise difference between the two, except that propositions are generally considered less important.

On your own: Go to https://arxiv.org/archive/math and pick a topic. Skim a few papers that are 30 pages or longer (that are not self-proclaimed "survey articles"), accounting only for how many theorems there are, how many propositions there are, and which ones are discussed in the introduction and how.

Lemmas:
Small but technical statements needed for larger proofs.
Usually not very interesting in their own right, but the interesting stuff can't be proven without them.
Use lemmas to break up otherwise very long proofs of propositions or theorems.

Definitions: A mathematical definition gives the meaning of a word (or phrase) in a specific way. The word (or phrase) is generally defined in terms of properties.

Example definitions from the book:

(i) An integer is *even* if it is the product of 2 and another integer.

(ii) An integer is *odd* if it is not even.

(iii) We call a set X with a finite number of elements a *finite set*.

(iv) If a natural number greater than 1 is divisible only by 1 and itself, then it is called *prime*.

(v) A prime number p is called a *twin prime* if $p-2$ or $p+2$ is prime.

(vi) A positive integer n is a *square number* if $n = x^2$ for some integer x.

(vii) A natural number is called *squarey* if its digits are the last digits of its square. (Non-standard)

(viii) A prime number is called *squarey-twinney* if it is a twin prime and is squarey. (Non-standard)

Stylistic note: We usually use \emph{} to highlight the word/phrase that's being defined. This helps for parsing the definition, as well as finding its definition later in the reading.

Socially acceptable misuse of "if/then".

Consider the two statements:

An integer is even if it is the product of 2 and another integer.

versus

The sum of two integers is even if both summands are even.

(For $a, b \in \mathbb{Z}$, we have $a + b$ is even if a and b are even.)

The first is a definition (of even).
The second is a theorem (or proposition, or whatever).

In the definition, an "if and only if" is *implied*. Technically, we mean "An integer is even if and only if it is the product of 2 and another integer." But by social convention, we almost never use "if and only if" in this context.

In the theorem, "if and only if" is definitely *not implied*. The implied $\Leftrightarrow$ is only acceptable in definitions.

How to read definitions

1. **Observe details.** Be mindful of orders of quantifiers; keep track of how many conditions are required; pay most attention to awkwardly worded conditions.
2. **Compare/contrast.** Is this built on top of a definition we already know? How is the similar to other definitions we've learned? How is it different?
3. **Examples.** Are there examples? List a few! Then, are there finitely many examples? Infinitely many? Trivial, standard, extreme, non-.

3. Examples. Are there examples? List a few! Then, are there finitely many examples? Infinitely many?

Standard examples: those which will help us remember the definition, and all its little quirks. Namely, easy enough to remember; hard enough to illuminate subtleties.

Trivial examples: The silliest, smallest examples that will help us quickly test understanding.

Common examples: Numbers: 0, 1; Sets: $\varnothing$.

Extreme examples: These are like trivial examples, in that they push our assumptions, e.g. numbers: 10,000,003; sets: the universal set.

Non-examples: Find things that almost fit the definition, but fail in one or two ways. Use non-examples to process subtleties in definitions.

Find examples and non-examples of each of the following. What would be a good trivial example? A non-trivial example?

1. Let X be a set. Define a binary operation on X to be a map $\star : X \times X \to X$. That is, $\star$ takes two elements of X and produces a third. (An example will include both a choice of set and a choice of map. For example, let X be the integers, and let $\star$ be multiplication.)
2. We say a binary operation on a set X is associative if for all $x, y, z \in X$, we have
$$x \star (y \star z) = (x \star y) \star z.$$
3. We say a binary operation on a set X is commutative if for all $x, y \in X$, we have
$$x \star y = y \star x.$$
4. Given a binary operation on a set X, an identity element of X (with respect to $\star$ is some $e \in X$ such that for all $x \in X$ we have
$$e \star x = x \quad \text{and} \quad x \star e = x.$$

How to read a theorem

1. Find the assumptions and conclusions
2. Rewrite in symbols or in words
3. Observe the detail
4. What does the theorem do and how it can be used?
5. Draw a picture
6. Apply to trivial examples and other extreme cases
7. Compare with earlier theorems
8. Rate the "strength" of the assumptions and conclusions
    - Is the converse true?
    - What happens to non-examples?
    - Can you generalize

Theorem 1: Let $f : X \to Y$ be a function between sets X and Y, and let $A, B \subseteq X$. Then $f(A \cup B) = f(A) \cup f(B)$.
Theorem 2: Let x be a natural number. If x is odd, then x^3 is odd.
Theorem 3: The square root of 2 is irrational.
Theorem 4: We have $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.