TECHNIQUES OF PROOF I: DIRECT METHOD; AND COMMON MISTAKES

1. Direct proofs about numbers

In these notes, we'll explore some examples of "direct" proofs. But first, we'll need a little more math.

Some information about integers. Let $n, d \in \mathbb{Z}$. We say d is a *divisor* of n if $n/d \in \mathbb{Z}$ (i.e. there exists some $k \in \mathbb{Z}$ for which $n = kd$). We say $p \in \mathbb{Z}$ is *prime* if for all $a, b \in \mathbb{Z}$, we have

$$\text{if } p \text{ is a divisor of } ab, \quad \text{then } p \text{ is a divisor of } a \text{ or } b.$$

In logical symbols, this is

$$\text{“}p \text{ prime} \equiv \forall a, b \in \mathbb{Z}\,(ab/p \in \mathbb{Z} \Rightarrow (a/p \in \mathbb{Z} \vee b/p \in \mathbb{Z}))\text{”}$$

This is equivalent to the only divisors of p being $\pm p$, ± 1.

Examples.

(a) The integer divisors of 6 are $1, 2, 3, 6, -1, -2, -3$, and -6. So 6 is not prime.
(b) 5 is prime since its only divisors are $1, 5, -1$, and -5.

We say a is *rational* if there exists $m, n \in \mathbb{Z}$ with $n \neq 0$ such that $a = m/n$. We say m/n is in *lowest terms* if m and n don't have any prime divisors in common.

Now on to our first example of a direct proof!

Theorem 1. *Suppose that $a \in \mathbb{Q}$ and $a^2 \in \mathbb{Z}$. Then $a \in \mathbb{Z}$.*

Unpacking the problem:

- Being rational means there are $m, n \in \mathbb{Z}$ with $n \neq 0$ such that $a = m/n$.
- We might as well assume m/n is in lowest terms, so that m and n have no common prime divisors.
- Being integral means that in lowest form, $m = \pm 1$.

Making a plan:

Start with "Let a be a rational number satisfying $a^2 \in \mathbb{Z}$."
Goal: Conclude $a \in \mathbb{Z}$.
To try: Let $m/n \in \mathbb{Z}$ such that $a = m/n$ and m/n is in lowest form. Compute a^2 and use $a^2 \in \mathbb{Z}$.

Proof. Let a be a rational number satisfying $a^2 \in \mathbb{Z}$. Since $a \in \mathbb{Q}$, there exists $m, n \in \mathbb{Z}$ (with $n \neq 0$) such that $a = m/n$. Assume, without loss of generality, that m/n is in lowest form (i.e. m and n have no common prime factors). Thus

$$a^2 = (m/n)^2 = m^2/n^2.$$

But since any prime factor of m^2 would also be a prime factor of m (and similarly for n^2 and n), we have m^2/n^2 is in lowest terms. So since $m^2/n^2 \in \mathbb{Z}$, we have $n^2 = 1$. So $n = \pm 1$. And thus $a = m/n \in \mathbb{Z}$, as desired. □

Theorem 2. *Let x and y be positive real numbers. If $y > x$, then $\frac{x+1}{y+1} > \frac{x}{y}$.*

Making a plan:

Start with "Let x and y be positive real numbers with $y > x$."
Goal: Conclude $\dfrac{x+1}{y+1} > \dfrac{x}{y}$.
To try: Start with $\dfrac{x+1}{y+1} > \dfrac{x}{y}$ and manipulate until I see how I can get there (scratch work).
Best if steps are "if and only if".

Scratch work: So long as $y > 0$, we have

$$\begin{aligned} \frac{x+1}{y+1} &> \frac{x}{y} && \text{if and only if} \\ (x+1)y &> x(y+1) && \text{if and only if} \\ xy + y &> xy + x && \text{if and only if} \\ y &> x, \end{aligned}$$

which is what we're assuming. So now all we have to do is turn this around!

Proof. Since $y > x$, we have

$$(x+1)y = xy + y > xy + x = (y+1)x.$$

Since $y > 0$, we have $y(y+1) > 0$. Thus, dividing both sides by $y(y+1)$ (which is non-zero since $y > 0$) gives

$$\frac{x+1}{y+1} > \frac{x}{y},$$

as desired. □

Warning! The book's proofs are occasionally more like first drafts, since they use a lot of symbols, and not enough words. Observe the difference between the "proof" of Thm 20.4 and this proof.

1.1. **Proving two numbers are equal.** If you want to show two numbers a and b are equal, you can either

1. prove $a \leq b$ and $b \leq a$; or
2. find a string of equivalent values $c_1, c_2, \ldots, c_\ell$ such that

$$a = c_1 = c_2 = \cdots = c_\ell = b.$$

2. Proving "if and only if"

Recall that "A if and only if B" is equivalent to $(A \Rightarrow B) \wedge (B \Rightarrow A)$. We also have

$$((A \Leftrightarrow B) \wedge (B \Leftrightarrow C)) \Rightarrow (A \Leftrightarrow C)$$

(you can check the truth table). Namely, a string of equivalencies proves and equivalency.

So to prove "A if and only if B", you can either

1. prove $A \Rightarrow B$ and $B \Rightarrow A$ (or any of the logically equivalent implications); or
2. find a string of equivalent statements $C_1, C_2, \ldots, C_\ell$ such that

$$A \Leftrightarrow C_1 \Leftrightarrow C_2 \Leftrightarrow \cdots \Leftrightarrow C_\ell \Leftrightarrow B.$$

3. Proofs about sets

3.1. **Proving subsets.** Let X and Y be sets. We say X is a *subset* of Y means

$$\text{for all } x \in X, \text{ we have } x \in Y.$$

This is

$$\forall x \in X (x \in Y), \quad \text{or, equivalently,} \quad x \in X \Rightarrow x \in Y.$$

To prove $X \subseteq Y$,

start with "let $x \in X$."
Goal: conclude $x \in Y$.

Theorem 3. *Let X and Y be sets. Then $X \cap Y \subseteq X \cup Y$.*

Proof. Let $x \in X \cap Y$. Then, by the definition of $X \cap Y$, we have $x \in X$ and $x \in Y$. In particular, $x \in X$, so that $(x \in X) \vee (x \in Y)$ holds. Thus $x \in X \cup Y$. Therefore $X \cap Y \subseteq X \cup Y$, as desired. □

3.2. **Proving two sets are equal.** Let X and Y be sets. We say X is *equal* to Y means

$$x \in X \text{ if and only if } x \in Y.$$

In logical symbols, this is

$$x \in X \Leftrightarrow x \in Y.$$

Theorem 4. *Let X, Y, and Z be sets. Then*

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z).$$

Proof. Let $a \in X \cap (Y \cup Z)$. This means

$$a \in X \quad \text{and} \quad (a \in Y) \vee (a \in Z).$$

We have shown that this is (logically) equivalent to

$$a \in X \wedge a \in Y \quad \text{or} \quad a \in X \wedge a \in Z.$$

By definition, this is equivalent to $a \in (X \cap Y) \cup (X \cap Z)$. In other words,

$$a \in X \cap (Y \cup Z) \quad \text{if and only if} \quad a \in (X \cap Y) \cup (X \cap Z).$$

So

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z),$$

as desired. □

One big tool for proving two sets are equal will be the following theorem.

Theorem 5. *Let X and Y be sets. Then $X = Y$ if and only if $X \subseteq Y$ and $X \supseteq Y$.*

Proof 1 (Using definitions and logical equivalencies). Suppose $X = Y$. This means

$$x \in X \text{ if and only if } x \in Y.$$

But, by "if and only if", we mean

$$x \in X \Rightarrow x \in Y \text{ and } x \in X \Leftarrow x \in Y.$$

By the definition of $\subseteq$, this ie equivalent to

$$X \subseteq Y \quad \text{and} \quad X \supseteq Y,$$

as desired. □

Proof 2 (Prove one direction at a time). Suppose $X = Y$. This means

$$x \in X \text{ if and only if } x \in Y.$$

Thus, if $x \in X$, then $x \in Y$, implying $X \subseteq Y$. Similarly, if $x \in Y$, then $x \in X$, implying $X \supseteq Y$.

Conversely, suppose that $X \subseteq Y$ and $X \supseteq Y$. Then if $x \in X$, we have $x \in Y$; and if $x \in Y$, then $x \in X$. So $x \in X$ if and only if $x \in Y$. Namely, $X = Y$. □

From now on, you can use this theorem to prove two sets are equal! To see what we mean, let's look at an example.

For each $n \in \mathbb{Z}_{>0}$, pick a subset $A_n \subseteq \mathbb{Z}$. Define

$$\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \cdots = \{x \in \mathbb{Z} \mid x \in A_n \text{ for all } n \in \mathbb{Z}_{>0}\}.$$

Example. Prove that if $A_n = \{1, 2, \ldots, n\}$, then

$$\bigcap_{n=1}^{\infty} A_n = \{1\}.$$

Using Theorem 5, we'll show that $\bigcap_{n=1}^{\infty} A_n \subseteq \{1\}$ and $\bigcap_{n=1}^{\infty} A_n \supseteq \{1\}$.

Proof. First, let $x \in \bigcap_{n=1}^{\infty} A_n$. Then since x is in every A_n, then in particular, $x \in A_1 = \{1\}$. So

$$\bigcap_{n=1}^{\infty} A_n \subseteq \{1\}.$$

Conversely, let $x \in \{1\}$, so that $x = 1$. Thus

$$x \in \{1, \ldots, n\} = A_n \quad \text{for all} \quad n \in \mathbb{Z}_{>0}.$$

Therefore $x \in \bigcap_{n=1}^{\infty} A_n$. So

$$\bigcap_{n=1}^{\infty} A_n \supseteq \{1\}.$$

Therefore, by Theorem 5, we have $\bigcap_{n=1}^{\infty} A_n = \{1\}$, as desired.

□

4. Summary of strategies

To prove "A ***if and only if*** B", you can either...

1. prove $A \Rightarrow B$ and $B \Rightarrow A$ (or any of the logically equivalent implications); or
2. find a string of equivalent statements $C_1, C_2, \ldots, C_\ell$ such that
$$A \Leftrightarrow C_1 \Leftrightarrow C_2 \Leftrightarrow \cdots \Leftrightarrow C_\ell \Leftrightarrow B.$$

To prove two ***numbers*** a ***and*** b ***are equal***, you can either...

1. prove $a \leq b$ and $b \leq a$; or
2. find a string of equivalent values $c_1, c_2, \ldots, c_\ell$ such that
$$a = c_1 = c_2 = \cdots = c_\ell = b.$$

To prove two ***sets*** X ***and*** Y ***are equal***, you can either...

1. prove $X \subseteq Y$ and $Y \subseteq X$; or
2. find a string sets $Z_1, Z_2, \ldots, Z_\ell$ such that
$$X = Z_1 = Z_2 = \cdots = Z_\ell = Y.$$

5. Common mistakes

5.1. **Don't assume what has to be proven.** Suppose you want to prove P. If you start with P, then all you can prove is $P \Rightarrow$ (stuff).

Example. Prove the following claim.

We have $-1 = 1$.

Non-proof. If $-1 = 1$, then
$$(-1)^2 = (1)^2, \qquad \text{so that} \qquad 1 = 1,$$
which is true. □

Here, we proved that "$-1 = 1 \Rightarrow 1 = 1$", which is true (F $\Rightarrow$ T is true). We did not show that $-1 = 1$.

Example. Prove the following claim.

If a and b are real numbers, then $a^2 + b^2 \geq 2ab$.

Non-proof. We have
$$a^2 + b^2 \geq 2ab \Rightarrow a^2 - 2ab + b^2 \geq 0 \Rightarrow (a-b)^2 \geq 0.$$
The last inequality is true, since the square of a number is always non-negative. So $a^2+b^2 \geq 2ab$. □

We wanted to show
$$a, b \in \mathbb{R} \quad \Rightarrow \quad a^2 + b^2 \geq 2ab.$$
What we actually showed was
$$(a, b \in \mathbb{R}) \wedge (a^2 + b^2 \geq 2ab) \quad \Rightarrow \quad (a-b)^2 \geq 0.$$

Fortunately, this proof can be fixed!

Proof. Since the square of a number is always non-negative, we have

$$0 \leq (a-b)^2 = a^2 - 2ab + b^2.$$

So, subtracting $2ab$ from both sides, we get

$$a^2 + b^2 \geq 2ab,$$

as desired. □

> As a **problem-solving strategy**, it's effective to assume the thing you want to show, and work backwards. This is your **scratch work**. But when you actually go to write the proof, you have to make sure you can work forwards!

5.2. **The dangers of square root.** When you're working with square root, you have to decide whether

(1) you're *undoing the function* $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ given by $x \mapsto x^2$; or
(2) you're working with the *function* $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $x \mapsto x^2$.

Namely, when computing the square root of 4, do you want

"2 and -2" or just "2".

By default, we usually define $\sqrt{x}$ as the *function*, returning the positive root only.

Example. Prove the following claim.

The solutions to $\sqrt{x+3} = x+1$ are given by $x = 1$ and $x = -2$.

Non-proof. If $\sqrt{x+3} = x+1$, then squaring both sides gives

$$x + 3 = (x+1)^2 = x^2 + 2x + 1.$$

So

$$0 = x^2 + x - 2 = (x-1)(x+2).$$

□

The problem arrises when we squared both sides: we threw in extra solutions! Namely, we found solutions to

$$\sqrt{x+3} = x+1 \quad \text{and} \quad -\sqrt{x+3} = x+1.$$

To see what I mean, try plugging in: at $x = 1$, we have

$$\sqrt{1+3} = \sqrt{4} = 2 \quad \text{and} \quad 1+1 = 2; \quad \checkmark$$

and at $x = -2$, we have

$$\sqrt{1+(-2)} = \sqrt{1} = 1 \quad \text{and} \quad 1+(-2) = -1. \quad ☹$$

5.3. **Don't divide by zero.**
Example. Prove that $1 = 2$.

Non-proof. Let $a = b$ be real numbers. Then, multiplying both sides by a, we get

$$a^2 = ab.$$

Subtracting b^2 from both sides gives

$$ab - b^2 = a^2 - b^2, \quad \text{so that} \quad b(a-b) = (a+b)(a-b).$$

Cancelling $(a-b)$, we get

$$b = a + b.$$

Therefore, since $a = b$, we can substitute back in to get

$$b = a + b = b + b = 2b. \quad \text{So, dividing by } b \text{ gives} \quad 1 = 2,$$

as desired. □

5.4. **Careful with negative signs.**
Example. Prove that $-1 > 1$.

Non-proof. Let $x = -1$. Then $x < 1$. So, since $a > b$ implies $1/a < 1/b$, we get

$$1/x > 1/1 = 1.$$

But since $1/x = 1/-1 = -1$, we have $-1 > 1$. □

Basically, when doing algebraic manipulation, be careful not to assume something is positive if that's not part of the claim. In particular, it's possible for $-x$ to be positive (if x was negative).

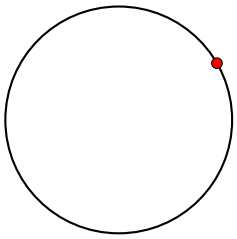
5.5. **Careful with using examples.** If you want to show that something exists, you can do so by giving an example. Similarly, if you want to show that a "for all" statement is *false* you can give a "counterexample" (we'll explore these soon). But if you want to show a "for all" statement to be *true*, examples won't do it for you!

As a **problem-solving strategy**, it's effective do some examples before you start to attempt a proof. This can help you understand the problem. However, once you move on to proving the statement, you must leave your examples behind!

Example. Draw n unique points on the circle, such that when you connect them with line segments, no more than two lines cross at any point. Prove that this will divide the circle into 2^{n-1} regions.

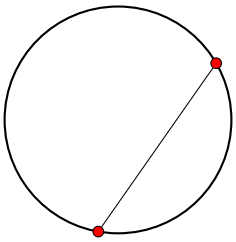
Non-proof. Let's do some examples:

$n = 1$:



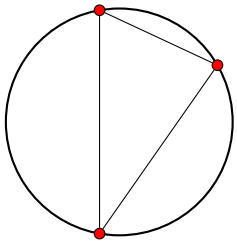
1 region

$n = 2$:



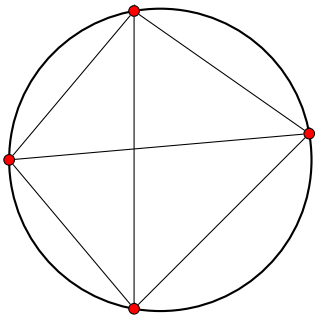
2 regions

$n = 3$:



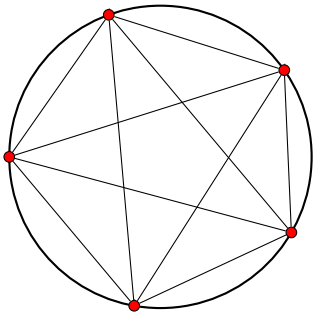
4 regions

$n = 4$:



8 regions

$n = 5$:

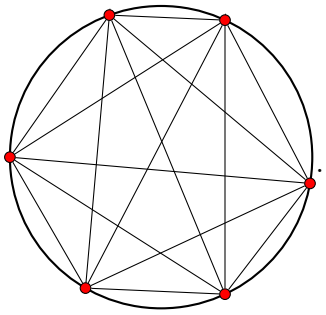


16 regions

Since this works for $n = 1, 2, 3, 4$, and 5, it must be true! □

Besides being a lacking argument, we "proved" a false statement. And here's where examples *do* work: we can show that the "for all" statement is false by giving *one* example where it fails (this demonstrating that not *all* examples will work):

$n = 6$:



31 regions!