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if p is a divisor of ab, then p is a divisor of a or b.

"p prime $\equiv \forall a, b \in \mathbb{Z}\,(ab/p \in \mathbb{Z} \Rightarrow (a/p \in \mathbb{Z} \vee b/p \in \mathbb{Z}))$"

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Theorem

Suppose that $a \in \mathbb{Q}$ *and* $a^2 \in \mathbb{Z}$*. Then* $a \in \mathbb{Z}$*.*

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Being integral means that in lowest form, $m = \pm 1$.

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Proof. Let a be a rational number satisfying $a^2 \in \mathbb{Z}$. Since $a \in \mathbb{Q}$, there exists $m, n \in \mathbb{Z}$ (with $n \neq 0$) such that $a = m/n$. Assume, without loss of generality, that m/n is in lowest form (i.e. m and n have no common prime factors).

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$$a^2 = (m/n)^2 = m^2/n^2.$$

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Proof. Since $y > x$, we have

$$(x+1)y = xy + y > xy + x = (y+1)x.$$

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Since $y > 0$, we have $y(y+1) > 0$.

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Since $y > 0$, we have $y(y+1) > 0$. Thus, dividing both sides by $y(y+1)$ gives

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as desired. □

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> **Warning!** The book's proofs are occasionally more like first drafts, since they use a lot of symbols, and not enough words. Observe the difference between the "proof" of Thm 20.4 and this proof.

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Recall that "A if and only if B" is equivalent to
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So to prove "A if and only if B", you can either...

1. prove $A \Rightarrow B$ and $B \Rightarrow A$ (or any of the logically equivalent implications); or
2. find a string of equivalent statements $C_1, C_2, \ldots, C_\ell$ such that

$$A \Leftrightarrow C_1 \Leftrightarrow C_2 \Leftrightarrow \cdots \Leftrightarrow C_\ell \Leftrightarrow B.$$

(We'll see examples of both of these.)

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(We'll see examples of both of these.)

Similarly, if you want to show two numbers a and b are equal, you can either...

1. prove $a \leqslant b$ and $b \leqslant a$; or
2. find a string of equivalent values $c_1, c_2, \ldots, c_\ell$ such that
$$a = c_1 = c_2 = \cdots = c_\ell = b.$$

Proving subsets

Defn. Let X and Y be sets. We say X is a subset of Y means

$$\text{for all } x \in X, \text{ we have } x \in Y.$$

This is

$$\forall x \in X (x \in Y), \quad \text{or, equivalently,} \quad x \in X \Rightarrow x \in Y.$$

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Let X and Y be sets. Then $X \cap Y \subseteq X \cup Y$.

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□

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$$x \in X \Rightarrow x \in Y \text{ and } x \in X \Leftarrow x \in Y.$$

By the definition of $\subseteq$, this ie equivalent to

$$X \subseteq Y \quad \text{and} \quad X \supseteq Y,$$

as desired.

$\square$

Proving two sets are equal

Theorem
Let X and Y be sets. Then $X = Y$ if and only if $X \subseteq Y$ and $X \supseteq Y$.

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From now on, you can use this theorem to prove two sets are equal!

Theorem

Let X and Y be sets. Then $X = Y$ if and only if $X \subseteq Y$ and $X \supseteq Y$.

For each $n \in \mathbb{Z}_{>0}$, pick a subset $A_n \subseteq \mathbb{Z}$. Define

$$\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \cdots = \{x \in \mathbb{Z} \mid x \in A_n \text{ for all } n \in \mathbb{Z}_{>0}\}.$$

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Ex. Prove that if $A_n = \{1, 2, \ldots, n\}$, then

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Conversely, let $x \in \{1\}$, so that $x = 1$. Thus $x \in \{1, \ldots, n\} = A_n$ for all $n \in \mathbb{Z}_{>0}$. Therefore $x \in \bigcap_{n=1}^{\infty} A_n$.

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Therefore $\bigcap_{n=1}^{\infty} A_n = \{1\}$, as desired.

Summary of strategies

To prove "A **if and only if** B", you can either...

1. prove $A \Rightarrow B$ and $B \Rightarrow A$ (or any of the logically equivalent implications); or
2. find a string of equivalent statements $C_1, C_2, \ldots, C_\ell$ such that
$$A \Leftrightarrow C_1 \Leftrightarrow C_2 \Leftrightarrow \cdots \Leftrightarrow C_\ell \Leftrightarrow B.$$

To prove two **numbers** a **and** b **are equal**, you can either...

1. prove $a \leqslant b$ and $b \leqslant a$; or
2. find a string of equivalent values $c_1, c_2, \ldots, c_\ell$ such that
$$a = c_1 = c_2 = \cdots = c_\ell = b.$$

To prove two **sets** X **and** Y **are equal**, you can either...

1. prove $X \subseteq Y$ and $Y \subseteq X$; or
2. find a string sets $Z_1, Z_2, \ldots, Z_\ell$ such that
$$X = Z_1 = Z_2 = \cdots = Z_\ell = Y.$$

Common mistakes

1. Assuming your desired conclusion.
2. Taking square roots badly.
3. Dividing by zero.
4. Forgetting things might be negative.
5. Using examples to deduce “for all” statements.

Error 1: Assuming your desired conclusion.

Claim: $-1 = 1$

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Non-proof.

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If $-1 = 1$, then

$$(-1)^2 = (1)^2, \qquad \text{so that} \qquad 1 = 1,$$

which is true. □

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If $-1 = 1$, then

$$(-1)^2 = (1)^2, \qquad \text{so that} \qquad 1 = 1,$$

which is true. □

What went wrong:

We proved that

$$\text{“}-1 = 1 \Rightarrow 1 = 1\text{”},$$

which is true (F $\Rightarrow$ T is true)!

We **did not** show that $-1 = 1$.

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Non-proof.

We have

$$a^2 + b^2 \geqslant 2ab \Rightarrow a^2 - 2ab + b^2 \geqslant 0$$

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Claim: If a and b are real numbers, then $a^2 + b^2 \geqslant 2ab$.

Non-proof.

We have

$$a^2 + b^2 \geqslant 2ab \Rightarrow a^2 - 2ab + b^2 \geqslant 0 \Rightarrow (a - b)^2 \geqslant 0.$$

Error 1: Assuming your desired conclusion.

Claim: If a and b are real numbers, then $a^2 + b^2 \geqslant 2ab$.

Non-proof.

We have

$$a^2 + b^2 \geqslant 2ab \Rightarrow a^2 - 2ab + b^2 \geqslant 0 \Rightarrow (a-b)^2 \geqslant 0.$$

The last inequality is true, since the square of a number is always non-negative. So $a^2 + b^2 \geqslant 2ab$.

$\square$

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We have

$$\boxed{a^2 + b^2 \geqslant 2ab \Rightarrow} \; a^2 - 2ab + b^2 \geqslant 0 \Rightarrow (a-b)^2 \geqslant 0.$$

The last inequality is true, since the square of a number is always non-negative. So $a^2 + b^2 \geqslant 2ab$.

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What went wrong:

We wanted to show

$$a, b \in \mathbb{R} \quad \Rightarrow \quad a^2 + b^2 \geqslant 2ab.$$

What we actually showed was

$$(a, b \in \mathbb{R}) \wedge (a^2 + b^2 \geqslant 2ab) \quad \Rightarrow \quad (a-b)^2 \geqslant 0.$$

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We have

$$\boxed{a^2 + b^2 \geqslant 2ab \Rightarrow} \; a^2 - 2ab + b^2 \geqslant 0 \Rightarrow (a-b)^2 \geqslant 0.$$

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Fortunately, we can fix this!

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We have

$$a^2 + b^2 \geqslant 2ab \Rightarrow a^2 - 2ab + b^2 \geqslant 0 \Rightarrow (a-b)^2 \geqslant 0.$$

The last inequality is true, since the square of a number is always non-negative. So $a^2 + b^2 \geqslant 2ab$.

□

Proof.

Since the square of a number is always non-negative, we have

$$0 \leqslant (a-b)^2$$

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□

Proof.

Since the square of a number is always non-negative, we have

$$0 \leqslant (a-b)^2 = a^2 - 2ab + b^2.$$

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Claim: If a and b are real numbers, then $a^2 + b^2 \geqslant 2ab$.

Non-proof.

We have

$$a^2 + b^2 \geqslant 2ab \Rightarrow a^2 - 2ab + b^2 \geqslant 0 \Rightarrow (a - b)^2 \geqslant 0.$$

The last inequality is true, since the square of a number is always non-negative. So $a^2 + b^2 \geqslant 2ab$.

□

Proof.

Since the square of a number is always non-negative, we have

$$0 \leqslant (a - b)^2 = a^2 - 2ab + b^2.$$

So, subtracting $2ab$ from both sides, we get

$$a^2 + b^2 \geqslant 2ab.$$

as desired.

□

Error 1: Assuming your desired conclusion.

As a **problem-solving strategy**, it's effective to assume the thing you want to show, and work backwards.

This is your **scratch work**.

But when you actually go to write the proof, you have to make sure you can work forwards!

Error 2: Taking square roots badly.

Our convention is to fix $\sqrt{x}$ to mean the *positive* root.
Namely, if $y^2 = x$, then $y = \pm\sqrt{x}$.

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Our convention is to fix $\sqrt{x}$ to mean the *positive* root. Namely, if $y^2 = x$, then $y = \pm\sqrt{x}$.

Claim. The solutions to $\sqrt{x+3} = x + 1$ are given by $x = 1$ and $x = -2$.

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Non-proof.

If $\sqrt{x+3} = x+1$, then squaring both sides gives

$$x + 3 = (x+1)^2$$

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If $\sqrt{x+3} = x + 1$, then squaring both sides gives

$$x + 3 = (x + 1)^2 = x^2 + 2x + 1.$$

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Non-proof.

If $\sqrt{x+3} = x+1$, then squaring both sides gives

$$x + 3 = (x+1)^2 = x^2 + 2x + 1.$$

So

$$0 = x^2 + x - 2$$

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Non-proof.
If $\sqrt{x+3} = x+1$, then squaring both sides gives

$$x + 3 = (x+1)^2 = x^2 + 2x + 1.$$

So

$$0 = x^2 + x - 2 = (x-1)(x+2).$$

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So $x = 1$ or $x = -2$. $\square$

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If $\sqrt{x+3} = x + 1$, then **squaring both sides** gives

$$x + 3 = (x + 1)^2 = x^2 + 2x + 1.$$

So

$$0 = x^2 + x - 2 = (x - 1)(x + 2).$$

So $x = 1$ or $x = -2$. □

What went wrong:

When we squared both sides, we threw in extra solutions!

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Non-proof.

If $\sqrt{x+3} = x + 1$, then **squaring both sides** gives

$$x + 3 = (x + 1)^2 = x^2 + 2x + 1.$$

So

$$0 = x^2 + x - 2 = (x - 1)(x + 2).$$

So $x = 1$ or $x = -2$. □

What went wrong:

When we squared both sides, we threw in extra solutions!
We actually found solutions to

$$\sqrt{x+3} = x + 1 \quad \text{and} \quad -\sqrt{x+3} = x + 1.$$

Error 3: Dividing by zero.

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Let $a = b$ be real numbers.

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Let $a = b$ be real numbers. Then, multiplying both sides by a, we get

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Let $a = b$ be real numbers. Then, multiplying both sides by a, we get

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Subtracting b^2 from both sides gives

$$ab - b^2 = a^2 - b^2$$

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Subtracting b^2 from both sides gives

$$ab - b^2 = a^2 - b^2, \quad \text{so that} \quad b(a - b) = (a + b)(a - b).$$

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Cancelling $(a - b)$, we get

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Therefore, since $a = b$, we can substitute back in to get

$$b = a + b = b + b = 2b. \quad \text{So, dividing by } b \text{ gives} \quad 1 = 2,$$

as desired. □

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Claim. $1 = 2$.

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Let $a = b$ be real numbers. Then, multiplying both sides by a, we get

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$$ab - b^2 = a^2 - b^2, \quad \text{so that} \quad b(a-b) = (a+b)(a-b).$$

Cancelling $(a-b)$, we get

$$b = a + b.$$

Therefore, since $a = b$, we can substitute back in to get

$$b = a + b = b + b = 2b. \quad \text{So, dividing by } b \text{ gives} \quad 1 = 2,$$

as desired. □

Error 4: Forgetting things might be negative

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Lesson: When doing algebraic manipulation, be careful not to assume things are positive (unless that's part of the assumptions). In particular, it's possible for $-x$ to be positive (if x was negative).

Error 5: Using examples badly

JUST WHEN YOU THOUGHT
YOU UNDERSTOOD THE PATTERN

$$\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin t}{t}\,\frac{\sin(t/101)}{t/101}\,dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin t}{t}\,\frac{\sin(t/101)}{t/101}\,\frac{\sin(t/201)}{t/201}\,dt = \frac{\pi}{2}$$

$$\int_0^\infty \frac{\sin t}{t}\,\frac{\sin(t/101)}{t/101}\,\frac{\sin(t/201)}{t/201}\,\frac{\sin(t/301)}{t/301}\,dt = \frac{\pi}{2}$$

and so on... but not forever! The formula

$$\int_0^\infty \frac{\sin t}{t}\,\frac{\sin(t/101)}{t/101}\,\frac{\sin(t/201)}{t/201}\cdots\frac{\sin(t/(100n+1))}{t/(100n+1)}\,dt = \frac{\pi}{2}$$

holds whenever $n < 9.8 \cdot 10^{42}$. But it eventually fails! It's *false* for all $n > 7.4 \cdot 10^{43}$.

See link to "Patterns that eventually fail" on course website.

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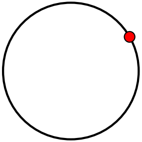
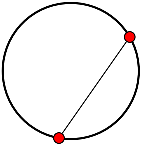
Non-proof. Let's do some examples:

$n = 1$:



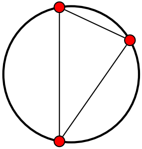
1 region

$n = 2$:



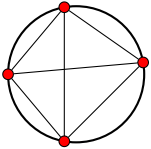
2 regions

$n = 3$:



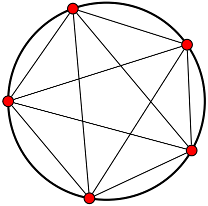
4 regions

$n = 4$:



8 regions

$n = 5$:



16 regions

Since this works for $n = 1, 2, 3, 4$, and 5, it must be true! □

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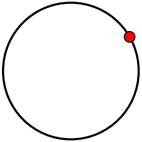
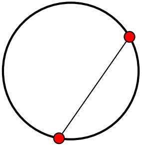


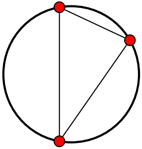
1 region

$n = 2$:



2 regions

$n = 3$:



4 regions

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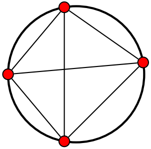

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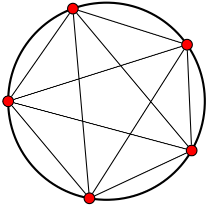
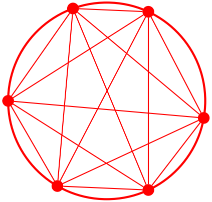


16 regions

$n = 6$:



31 regions!

Since this works for $n = 1, 2, 3, 4$, and 5, it must be true! □