Quantifiers – "For all" $\forall$ and "There exists" $\exists$

A quantifier is a phrase that tells you how many objects you're talking about. Good for pinning down conditional statements.

- For every real number $x \in \mathbb{R}$, we have x^2 is non-negative.
- For $x = -1$ and 1, the function $f(x) = x^4 - 2x^2$ is minimal.
- The equation $x^2 + 1 = 0$ has no real solutions.
- There is at least one real solution to $x^5 + 2x - 1 = 0$.

The phrase "for all", denoted $\forall$, is a universal quantifier.

LaTeX code: \forall

The phrase "there exists", denoted $\exists$ is an existential quantifier.

LaTeX code: \exists

Note that "for all" is *really* strong, and implies "there exists".
Examples:

1. For all $x \in \mathbb{R}$, we have $x^2 \geqslant 0$.
2. For all polynomials $p(x)$ of odd degree, $p(x) = 0$ has at least one real solution.
3. For all finite subsets $S \subset \mathbb{R}$, S has a maximal element.

On the other end "there exists" is really weak, and doesn't even always provide an example.

1. There exists $x \in \mathbb{R}$ such that $x^2 = 1$.
2. For all polynomials $p(x)$ of odd degree, there exists $x \in \mathbb{R}$ such that $p(x) = 0$.
3. For all finite subsets $S \subset \mathbb{R}$, there exists $s \in S$ such that $s \geqslant t$ for all $t \in S$.

Combining quantifiers: order matters!

Example:

For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $y > x$.

In plain language, this says

"Given any integer x, you can always find a bigger integer y."

True!

Example:

There exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $y > x$.

In plain language, this says

"There's some integer y that's bigger than all other integers."

False!

Exception: The order of consecutive $\forall$'s are interchangeable, and the order of consecutive $\exists$'s are interchangeable.

Notational format:

quantifiers(conditional statement)

For all $x \in \mathbb{R}$, we have $x^2 \geqslant 0$.	$\forall x \in \mathbb{R}(x^2 \geqslant 0)$
There exists $x \in \mathbb{R}$ such that $x^2 = 1$.	$\exists x \in \mathbb{R}(x^2 = 1)$
For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $y > x$.	$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}(y > x)$
There exists $y \in \mathbb{Z}$ such that for all $y \in \mathbb{R}$, we have $x^2 > y$.	$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}(x^2 > y)$

You try:

Put each of the following into words, and decide whether it's true or false.

(i) $\exists x \in \mathbb{R}(x^2 = x)$, (ii) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}(x^2 = y)$,

(iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}(x^2 = y)$

Put each of the following into symbols.

(iv) The function $f(x) = x^2 - 2$ has a minimum value.

(v) Every animal eats some kind of food.

Note: "for all" statements can be rewritten as implications.

Examples:

"For all $x \in \mathbb{R}$, we have $x^2 \geqslant 0$." $\qquad \forall x \in \mathbb{R}(x^2 \geqslant 0)$

is equivalent to

"If $x \in \mathbb{R}$ then $x^2 \geqslant 0$." $\qquad x \in \mathbb{R} \Rightarrow x^2 \geqslant 0$

"For all finite subsets $S \subset \mathbb{R}$, S has a maximal element."

is equivalent to

"If S is a finite subset of $\mathbb{R}$, then S has a maximal element."

Warning: It can be easier to stack quantifiers than to stack implications and $\exists$'s.

Negating quantifiers

To negate "quantifiers(conditional statement)",

1. change all $\forall$'s to $\exists$, and vice versa;
2. negate the conditional statement.

Examples:

$$\neg(\exists x \in \mathbb{R}(x^2 = x)) \quad \text{is equiv. to} \quad \forall x \in \mathbb{R}(x^2 \neq x)$$

In other words, the negation of

"$x^2 = x$ for some real number x"

is

"$x^2 \neq x$ for all real x".

Examples:

$$\neg(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}(x^2 = y)) \quad \text{is equiv. to} \quad \forall x \in \mathbb{R}, \exists y \in \mathbb{R}(x^2 \neq y)$$

In other words, the negation of

"there's some real x for which $y^2 = x$ for all real y"

is

"for every real x, there's a real y that's not equal to x^2".

Proof techniques

Implications: Recall $A \Rightarrow B$ can only be false if A is true and B is false. So to show $A \Rightarrow B$, start by assuming A.

Contrapositive: $A \Rightarrow B$ is equivalent to $\neg B \Rightarrow \neg A$. To show $\neg B \Rightarrow \neg A$, start by assuming B.

To show for all, start with a "generic" example.

Proof techniques

To show for all, start with a "generic" example.

Example: For every negative real x, we have $x^2 > x$.
In symbols, this is

$$\forall x \in \mathbb{R}_{<0}(x^2 > x).$$

To prove "$\forall x \in \mathbb{R}_{<0}(x^2 > x)$"...
Start with: "Let x be a negative real number."
Goal: Conclude $x^2 > x$.

Alternatively, $\forall x \in \mathbb{R}_{<0}(x^2 > x)$ is equivalent to $x \in \mathbb{R}_{<0} \Rightarrow x^2 > x$.
To prove "$x \in \mathbb{R}_{<0} \Rightarrow x^2 > x$"...
Start with: "Assume x is a negative real number."
Goal: Conclude $x^2 > x$.

Or, we can use the contrapositive!
To prove "$x^2 \not> x \Rightarrow x \notin \mathbb{R}_{<0}$"...
Start with: "Assume $x^2 \not> x$."
Goal: Conclude $x \notin \mathbb{R}_{<0}$.

Proof techniques

To show there exists directly, give an example.

Example: There exists $x \in \mathbb{R}$ such that $x^2 = x$.
In symbols, this is

$$\exists x \in \mathbb{R}(x^2 = x).$$

To prove "$\exists x \in \mathbb{R}(x^2 = x)$"...
Find an example: Fine one solution to $x^2 = x$.

Alternatively, you can prove the negation is false.
We have "$\neg(\exists x \in \mathbb{R}(x^2 = x))$" is equivalent to $\forall x \in \mathbb{R}(x^2 \neq x$.
Go back to techniques for proving "for all"...

Yikes! It's usually easier to find one example than to deal with *all* possible examples at once. However, to show a "there exists" is false, move to proving the negation is true.

Proving things false

If A is a statement, then

$$A \text{ is false} \quad \text{if and only if} \quad \neg A \text{ is true.}$$

Example: Show the following is false...

"For all $x \in \mathbb{R}$, we have $x^2 \geqslant x$."

In symbols, this is

$$\forall x \in \mathbb{R}(x^2 \geqslant x).$$

The negation of this is

$$\exists x \in \mathbb{R}(x^2 \ngeqslant x).$$

To show "$\exists x \in \mathbb{R}(x^2 \ngeqslant x)$" is <u>true</u>, give an example.

Proof. We have the statement is false since for $x = 1$, we have $x^2 = 1^2 = 1 \ngeqslant 1 = x$. □

Proving things false

If A is a statement, then

$$A \text{ is false} \quad \text{if and only if} \quad \neg A \text{ is true.}$$

Example: Show the following is false...

"There exists $x \in \mathbb{R}$ such that $x^2 + 1 = 0$."

In symbols, this is

$$\exists x \in \mathbb{R}(x^2 + 1 = 0).$$

The negation of this is

$$\forall x \in \mathbb{R}(x^2 + 1 \neq 0).$$

To show "$\forall x \in \mathbb{R}(x^2 + 1 \neq 0)$" is <u>true</u>...
Start with: "Let x be a real number."
Goal: Conclude $x^2 + 1 \neq 0$.

Proof. Let x be a real number.
Then $x^2 \geqslant 0$. So $x^2 + 1 \geqslant 1 > 0$. Thus $x^2 \neq 0$.
Therefore there does not exist a real solution to $x^2 + 1 = 0$. □

Back to combining quantifiers examples. . .

Example:

For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $y > x$.

In symbols, this is

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}(y > x).$$

Plan:

- $*$ To prove "$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}(y > x)$"...
  Start with: "Let x be an integer."
  Goal: Conclude $\exists y \in \mathbb{Z}(y > x)$.
- $*$ To prove "$\exists y \in \mathbb{Z}(y > x)$"...
  Find example: Find a $y \in \mathbb{Z}$ that's bigger than x.

Proof.

Let x be an integer.

Then $y = x + 1$ is also an integer and $y > x$. □

Back to combining quantifiers examples. . .

Example: There exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $y > x$.

In symbols, this is

$$\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}(y > x).$$

(Remember, we would like to show this is false, which is the same as showing that the negation is true!)

The negation of $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}(y > x)$ is

$$\text{“}\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}(y \not> x).\text{”}$$

Plan:

- $*$ To prove "$\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z}(y \not> x)$"...
  Start with: "Let y be an integer."
  Goal: Conclude $\exists x \in \mathbb{Z}(y \not> x)$.
- $*$ To prove "$\exists x \in \mathbb{Z}(y \not> x)$"...
  Find example: Find a $x \in \mathbb{Z}$ that's not bigger than y.

Proof.

Let y be an integer.

Then $x = y$ is also an integer satisfying $y \not> x$.

Thus there is no integer y for which $y > x$ for all $x \in \mathbb{Z}$. □

You try:

For each of the following,

(a) Rewrite the statement in symbols.

(b) Negate the statement.

(c) Rewrite the negation in words.

(d) Decide whether you think the statement is true or false.

(e) Devise a plan to prove or disprove the statement.

1. For all $x \in \mathbb{Z}$, we have $x^2 + 1$ is odd.
2. For all $x \in \mathbb{R}_{>0}$, there exists an $n \in \mathbb{Z}_{>0}$ such that $1/n < x$.
3. For all $n \in \mathbb{Z}_{>0}$, we have $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$
4. There exists $n \in \mathbb{Z}_{>0}$ such that $n(n+1)$ is odd.