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Note that “for all” is *really* strong, and implies “there exists”.

Examples:

1. For all $x \in \mathbb{R}$, we have $x^2 \geqslant 0$.
2. For all polynomials $p(x)$ of odd degree, $p(x) = 0$ has at least one real solution.
3. For all finite subsets $S \subset \mathbb{R}$, S has a maximal element.

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On the other end “there exists” is really weak, and doesn't even always provide an example.

1. There exists $x \in \mathbb{R}$ such that $x^2 = 1$.
2. For all polynomials $p(x)$ of odd degree, there exists $x \in \mathbb{R}$ such that $p(x) = 0$.
3. For all finite subsets $S \subset \mathbb{R}$, there exists $s \in S$ such that $s \geqslant t$ for all $t \in S$.

Combining quantifiers: order matters!

Example:

For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $y > x$.

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In plain language, this says

"Given any integer x, you can always find a bigger integer y."

True!

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False!

Exception: The order of consecutive $\forall$'s are interchangeable, and the order of consecutive $\exists$'s are interchangeable.

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For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $y > x$.	$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}(y > x)$

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There exists $y \in \mathbb{Z}$ such that for all $y \in \mathbb{R}$, we have $x^2 > y$.	$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}(x^2 > y)$

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There exists $y \in \mathbb{Z}$ such that for all $y \in \mathbb{R}$, we have $x^2 > y$.	$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}(x^2 > y)$

You try:

Put each of the following into words, and decide whether it's true or false.

(i) $\exists x \in \mathbb{R}(x^2 = x)$, (ii) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}(x^2 = y)$,

(iii) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}(x^2 = y)$

Put each of the following into symbols.

(iv) The function $f(x) = x^2 - 2$ has a minimum value.

(v) Every animal eats some kind of food.

Note: "for all" statements can be rewritten as implications.

Examples:

"For all $x \in \mathbb{R}$, we have $x^2 \geqslant 0$." $\forall x \in \mathbb{R}(x^2 \geqslant 0)$

is equivalent to

"If $x \in \mathbb{R}$ then $x^2 \geqslant 0$." $x \in \mathbb{R} \Rightarrow x^2 \geqslant 0$

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"For all finite subsets $S \subset \mathbb{R}$, S has a maximal element."

is equivalent to

"If S is a finite subset of $\mathbb{R}$, then S has a maximal element."

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Warning: It can be easier to stack quantifiers than to stack implications and $\exists$'s.

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In other words, the negation of

"there's some real x for which $y^2 = x$ for all real y"

is

"for every real x, there's a real y that's not equal to x^2".

Proof techniques

Implications: Recall $A \Rightarrow B$ can only be false if A is true and B is false. So to show $A \Rightarrow B$, start by assuming A.

Contrapositive: $A \Rightarrow B$ is equivalent to $\neg B \Rightarrow \neg A$. To show $\neg B \Rightarrow \neg A$, start by assuming B.

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Start with: "Let x be a negative real number."

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To prove "$x \in \mathbb{R}_{<0} \Rightarrow x^2 > x$"...
Start with: "Assume x is a negative real number."
Goal: Conclude $x^2 > x$.

Or, we can use the contrapositive!
To prove "$x^2 \not> x \Rightarrow x \notin \mathbb{R}_{<0}$"...
Start with: "Assume $x^2 \not> x$."
Goal: Conclude $x \notin \mathbb{R}_{<0}$.

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Alternatively, you can prove the negation is false.

We have "$\neg(\exists x \in \mathbb{R}(x^2 = x))$" is equivalent to $\forall x \in \mathbb{R}(x^2 \neq x$.

Go back to techniques for proving "for all"...

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Yikes! It's usually easier to find one example than to deal with *all* possible examples at once. However, to show a "there exists" is false, move to proving the negation is true.

Proving things false

If A is a statement, then

$$A \text{ is false} \quad \text{if and only if} \quad \neg A \text{ is true.}$$

Example: Show the following is false...

"For all $x \in \mathbb{R}$, we have $x^2 \geqslant x$."

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To show "$\exists x \in \mathbb{R}(x^2 \ngeqslant x)$" is <u>true</u>, give an example.

Proof. We have the statement is false since for $x = 1$, we have $x^2 = 1^2 = 1 \ngeqslant 1 = x$. □

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Start with: "Let x be a real number."
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Proof. Let x be a real number.

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Proof. Let x be a real number.
Then $x^2 \geqslant 0$.

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Proof. Let x be a real number.
Then $x^2 \geqslant 0$. So $x^2 + 1 \geqslant 1 > 0$.

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  Goal: Conclude $x^2 + 1 \neq 0$.

Proof. Let x be a real number.
Then $x^2 \geqslant 0$. So $x^2 + 1 \geqslant 1 > 0$. Thus $x^2 \neq 0$.

Proving things false

If A is a statement, then

$$A \text{ is false} \quad \text{if and only if} \quad \neg A \text{ is true.}$$

Example: Show the following is false...

"There exists $x \in \mathbb{R}$ such that $x^2 + 1 = 0$."

In symbols, this is

$$\exists x \in \mathbb{R}(x^2 + 1 = 0).$$

The negation of this is

$$\forall x \in \mathbb{R}(x^2 + 1 \neq 0).$$

To show "$\forall x \in \mathbb{R}(x^2 + 1 \neq 0)$" is <u>true</u>...

Start with: "Let x be a real number."

Goal: Conclude $x^2 + 1 \neq 0$.

Proof. Let x be a real number.
Then $x^2 \geqslant 0$. So $x^2 + 1 \geqslant 1 > 0$. Thus $x^2 \neq 0$.
Therefore there does not exist a real solution to $x^2 + 1 = 0$. □

Back to combining quantifiers examples...

Example:

For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $y > x$.

Back to combining quantifiers examples...

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For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $y > x$.

In symbols, this is

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z} (y > x).$$

Back to combining quantifiers examples...

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In symbols, this is

$$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}(y > x).$$

Plan:

* To prove "$\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}(y > x)$"...
  Start with: "Let x be an integer."
  Goal: Conclude $\exists y \in \mathbb{Z}(y > x)$.

Back to combining quantifiers examples...

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Back to combining quantifiers examples...

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Back to combining quantifiers examples...

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Proof.

Let x be an integer.

Then $y = x + 1$ is also an integer and $y > x$. □

Back to combining quantifiers examples...

Example: There exists $y \in \mathbb{Z}$ such that for all $x \in \mathbb{Z}$, $y > x$.

Back to combining quantifiers examples. . .

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The negation of $\exists y \in \mathbb{Z}, \forall x \in \mathbb{Z}(y > x)$ is

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Proof.
Let y be an integer.

Back to combining quantifiers examples. . .

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* To prove "$\exists x \in \mathbb{Z}(y \not> x)$"...
  Find example: Find a $x \in \mathbb{Z}$ that's not bigger than y.

Proof.

Let y be an integer.
Then $x = y$ is also an integer satisfying $y \not> x$.
Thus there is no integer y for which $y > x$ for all $x \in \mathbb{Z}$. □

You try:

For each of the following,

(a) Rewrite the statement in symbols.

(b) Negate the statement.

(c) Rewrite the negation in words.

(d) Decide whether you think the statement is true or false.

(e) Devise a plan to prove or disprove the statement.

1. For all $x \in \mathbb{Z}$, we have $x^2 + 1$ is odd.
2. For all $x \in \mathbb{R}_{>0}$, there exists an $n \in \mathbb{Z}_{>0}$ such that $1/n < x$.
3. For all $n \in \mathbb{Z}_{>0}$, we have $\sqrt[n]{n!} < \sqrt[n+1]{(n+1)!}$
4. There exists $n \in \mathbb{Z}_{>0}$ such that $n(n+1)$ is odd.