Last time:

An implication is a statement of the form

"If statement A is true, then statement B is true."

An implication "$A \Rightarrow B$" is false when A is true and B is false, and is true otherwise. This is equivalent to $(\neg A) \vee B$.

A	B	$A \Rightarrow B$	$(\neg A) \vee B$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Negating implications

Recall that

$$\neg(A \vee B) \quad \text{is equivalent to} \quad (\neg A) \wedge (\neg B),$$

and

$$A \Rightarrow B \quad \text{is equivalent to} \quad (\neg A) \vee B.$$

So

$$\begin{aligned} \neg(A \Rightarrow B) \quad & \text{is equivalent to} \quad \neg((\neg A) \vee B), \\ & \text{which is equiv. to} \quad \neg(\neg A) \wedge \neg B, \\ & \text{which is equiv. to} \quad A \wedge \neg B. \end{aligned}$$

For example, the negation of

"If Jill's enrolled in this class, then Jill's a student at CUNY"

is

"Jill is enrolled in this class and Jill is not a student at CUNY."

Claim: The $\neg(A \Rightarrow B)$ is equivalent to $A \wedge \neg B$.

A	B	$A \Rightarrow B$	$\neg(A \Rightarrow B)$	$\neg B$	$A \wedge \neg B$
T	T				
T	F				
F	T				
F	F				

Messing with implications. . .

Consider the implication $A \Rightarrow B$

"*If it's raining, then the street is wet.*"

Negation: $\neg(A \Rightarrow B)$, which is equivalent to $(\neg A) \wedge B$.
Ex: "*It's raining, and the street is not wet.*"

Inverse: $(\neg A) \Rightarrow (\neg B)$
Ex: "*If it's not raining, then the street is not wet.*"

Converse: $B \Rightarrow A$
Ex: "*If the street is wet, then it's raining.*"

Contrapositive: $\neg B \Rightarrow \neg A$
Ex: "*If the street is not wet, then it's not raining.*"

Consider the implication $A \Rightarrow B$

"If it's raining, then the street is wet."

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A	B	$A \Rightarrow B$	Negation	Inverse	Converse	Contrapositive
T	T	T	F	T	T	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	F	T	T	T

Negation: $\neg(A \Rightarrow B)$, which is equivalent to $(\neg A) \wedge B$.

Inverse: $(\neg A) \Rightarrow (\neg B)$

Converse: $B \Rightarrow A$

Contrapositive: $\neg B \Rightarrow \neg A$

A	B	$A \Rightarrow B$	Negation	Inverse	Converse	Contrapositive
T	T	T	F	T	T	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	F	T	T	T

To show $A \Rightarrow B$ is **true** (respectively **false**), you can show

- the negation is **false** (respectively **true**); or
- the contrapositive is **true** (respectively **false**).

To show the converse $B \Rightarrow A$ is **true**, you can instead show the inverse is **true**.

In general, the truth of the inverse has nothing to do with the truth of the negation!

Negation: $\neg(A \Rightarrow B)$
Inverse: $(\neg A) \Rightarrow (\neg B)$
Converse: $B \Rightarrow A$
Contrapositive: $\neg B \Rightarrow \neg A$

You try: For each of the following
(a) rewrite the statement as "if ..., then ..."; and
(b) write, in English, (i) the negation, (ii) the inverse,
(iii) the converse, (iv) the contrapositive.

1. I wear a coat whenever it's cold outside.

2. The square of any real number is positive.

3. Consider an object in an inertial frame of reference. That object will continue to move at a constant velocity unless acted upon by a force.

Necessary and sufficient conditions

A necessary condition is one which must hold for a conclusion to be true. It does not guarantee that the result is true.

Ex: In order to get an A in this class, you must do the homework.

Ex: Let $a, b \in \mathbb{Z}$.

In order for ab to be odd, it is necessary for a to be odd.

$*$ If N is necessary for A, then $A \Rightarrow N$.

A sufficient condition is one which guarantees the conclusion is true. The conclusion may be true even if the condition is not satisfied.

Ex: In order to get an A in this class,

it is sufficient to get 100% on every assignment and exam.

Ex: Let $a, b \in \mathbb{Z}$. In order for ab to be odd,

it is sufficient for a and b to be equal to 3.

$*$ If S is sufficient for A, then $S \Rightarrow A$.

Necessary and sufficient conditions

A necessary condition is one which must hold for a conclusion to be true. (If N is necessary for A, then $A \Rightarrow N$.)

A sufficient condition is one which guarantees the conclusion is true. (If S is sufficient for A, then $S \Rightarrow A$.)

Note: Sufficient conditions imply necessary conditions!

(If $S \Rightarrow A$ and $A \Rightarrow N$, then $S \Rightarrow N$.)

Example: Let $a, b \in \mathbb{Z}$. In order for ab to be odd (A is "ab is odd"),

it is necessary for a to be odd (N is "a is odd"), and

it is sufficient for a and b to be equal to 3 (S is "$a = b = 3$").

Note that "$a = b = 3$" implies "a is odd" ($S \Rightarrow N$).

A necessary and sufficient conditions is a condition that is both necessary and sufficient.

Ex: Let $a, b \in \mathbb{Z}$. In order for ab to be odd,

it is necessary and sufficient for a and b to both be odd.

Neg: $\neg(A \Rightarrow B)$ Inv: $(\neg A) \Rightarrow (\neg B)$ Conv: $B \Rightarrow A$ Contr: $\neg B \Rightarrow \neg A$

You try: For each of the following
(a) rewrite the statement as "if ..., then ..."; and
(b) write, in English, (i) the negation, (ii) the inverse, (iii) the converse, (iv) the contrapositive.

1. It's necessary to be at least 18 years old to vote in the US.

2. It's sufficient for water to be 120° Celsius to boil.

3. To get into a math Ph.D. program, you have to take the GRE.

For each of the following, give (a) necessary condition that's not sufficient, (b) sufficient condition that's not necessary, (b) a (set of) condition(s) that's both necessary and sufficient.

1. $ab > 0$.
2. $\cos(\theta) = \sqrt{2}/2$.
3. $a + b$ is an even integer.

Logical equivalence, “if and only if”

We say A and B are logically equivalent statements if $A \Rightarrow B$ and $B \Rightarrow A$.
When A and B are equivalent, we say A if and only if B, written $A \Leftrightarrow B$.

Namely,

$$A \Leftrightarrow B \text{ means } (A \Rightarrow B) \wedge (B \Rightarrow A).$$

A	B	$A \Rightarrow B$	$A \Leftarrow B$	$A \Leftrightarrow B$	$(A \Rightarrow B) \wedge (\neg A \Rightarrow \neg B)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Let a and b be integers.
Example: We have $a > b$ if and only if $a - b > 0$.
Example: We have a is even if and only if $a + 2$ is even.
Example: We have a is even if and only if a^2 is even.