Last time:

An implication is a statement of the form

"If statement A is true, then statement B is true."

An implication "$A \Rightarrow B$" is false when A is true and B is false, and is true otherwise. This is equivalent to $(\neg A) \vee B$.

A	B	$A \Rightarrow B$	$(\neg A) \vee B$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Negating implications

Recall that

$$\neg(A \vee B) \quad \text{is equivalent to} \quad (\neg A) \wedge (\neg B),$$

and

$$A \Rightarrow B \quad \text{is equivalent to} \quad (\neg A) \vee B.$$

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For example, the negation of

"If Jill's enrolled in this class, then Jill's a student at CUNY"

is

"Jill is enrolled in this class and Jill is not a student at CUNY."

Claim: The $\neg(A \Rightarrow B)$ is equivalent to $A \wedge \neg B$.

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T	T				
T	F				
F	T				
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Messing with implications. . .

Negation: $\neg(A \Rightarrow B)$, which is equivalent to $(\neg A) \wedge B$.

Messing with implications...

Consider the implication $A \Rightarrow B$

"*If it's raining, then the street is wet.*"

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Ex: "*It's raining, and the street is not wet.*"

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To show $A \Rightarrow B$ is **true**, you can instead show

- the negation is **false** ; or
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In general, the truth of the inverse has nothing to do with the truth of the negation!

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You try: For each of the following
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1. I wear a coat whenever it's cold outside.

2. The square of any real number is positive.

3. Consider an object in an inertial frame of reference. That object will continue to move at a constant velocity unless acted upon by a force.

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1. I wear a coat whenever it's cold outside.

   If it's cold outside, then I wear a coat.

2. The square of any real number is positive.

   If a is a real number, then x^2 is positive.

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   If that object is not acted upon by a force, then it will continue to move at a constant velocity.

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it is sufficient for a and b to be equal to 3.

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A necessary condition is one which must hold for a conclusion to be true. (If N is necessary for A, then $A \Rightarrow N$.)

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Example: Let $a, b \in \mathbb{Z}$. In order for ab to be odd (A is "ab is odd"),
it is necessary for a to be odd (N is "a is odd"), and
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(a) rewrite the statement as "if ..., then ..."; and

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1. It's necessary to be at least 18 years old to vote in the US.

2. It's sufficient for water to be 120° Celsius to boil.

3. To get into a math Ph.D. program, you have to take the GRE.

For each of the following, give (a) necessary condition that's not sufficient, (b) sufficient condition that's not necessary, (b) a (set of) condition(s) that's both necessary and sufficient.

1. $ab > 0$.
2. $\cos(\theta) = \sqrt{2}/2$.
3. $a + b$ is an even integer.

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   If a person can vote in the US, then they're at least 18 years old.
2. It's sufficient for water to be 120° Celsius to boil.
   If water is 120 ° Celsius, then it's boiling.
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Let a and b be integers.
Example: We have $a > b$ if and only if $a - b > 0$.

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