

A **statement** is a sentence that is either always true or always false.

Examples

- ▶ $2 + 2 = 4$.
- ▶ Suppose $x^2 = 2$. Then x is not an integer.
- ▶ All cats are gray.
- ▶ I have a gray cat.

Non-examples

- ▶ x is an odd number.
- ▶ Let “Spot” be a cat. Then Spot is gray.
- ▶ This statement is false.

A **conditional statement** is almost a statement—it *would* be a statement if we added enough extra information. For example

“ x is not an integer.”

versus

“Suppose $x^2 = 2$. Then x is not an integer.”

From the reading:

Which of the following are statements?

- (i) Aristotle was Greek.
- (ii) Aristotle was great.
- (iii) The number $\sqrt{2}$ is rational.
- (iv) The square root of an integer is a rational number.
- (v) $3x^2 + 20x - 5 = 0$.
- (vi) Let x be an integer. Then \sqrt{x} is rational.
- (vii) There is a number x such that $\sin(x) = x$.
- (viii) There is an infinite number of natural numbers.
- (ix) There is an infinite number of rational numbers.

Precision is paramount!

From the reading:

Imagine that there are coins on a table; you count them and find there are exactly three of them—no more, no fewer. Which of the statements are true?

- (i) There are four coins on the table.
- (ii) There are two coins on the table.
- (iii) There are three coins on the table.
- (iv) There is a coin on the table.

Negations

The **negation** of statement A is the statement that is false exactly when A is true, and vice versa.

Example: The negation of
“It is raining outside” is “It is not raining outside.”

Example: The negation of
“ x is odd” is “ x is even.”

Example: The negation of
“All cats are gray” is “Not all cats are gray.”

Compare this to

“All cats are not grey.”

Problem: it is possible for “All cats are gray” and “All cats are not grey” to both be false at the same time.

Notation: If A is a statement,
then we write its negation as $\neg A$ or $\text{not}(A)$, read “not A ”. (\LaTeX :
 \neg is `\neg`)

Building new statements from old

Let A and B be statements.

- ▶ $\neg A$ is the negation of A , which is true if and only if A is false.
- ▶ $A \wedge B$, read “ A and B ”, is true whenever both A and B are both true. (L^AT_EX: \wedge is `\land`)
- ▶ $A \vee B$, read “ A or B ”, is true whenever A is true or B is true, or both. (**inclusive**) (L^AT_EX: \vee is `\lor`)

Example. Let A be the (conditional) statement “ x is even” and B be the (conditional) statement “ x is a multiple of 3”. Then

- ▶ $\neg A$ is “ x is not a multiple of 2”;
- ▶ $A \wedge B$ is “ x is a multiple of 6”; and
- ▶ $A \vee B$ is “ x is a multiple of 2 and/or 3.”

Notation: \wedge and \vee mimic \cap and \cup . Namely, if \mathcal{A} (resp. \mathcal{B}) is the set of things satisfying a conditional statement A (resp. B), then

$\mathcal{A} \cap \mathcal{B}$ is the set of things satisfying $A \wedge B$,

and

$\mathcal{A} \cup \mathcal{B}$ is the set of things satisfying $A \vee B$.

Truth tables

Let A and B be statements.

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A **truth table** is just a way of summarizing all the possibilities for statements being true or false.

A	B	$\neg A$	$\neg B$	$A \wedge B$	$A \vee B$
T	T				
T	F				
F	T				
F	F				

Two statements are **equivalent statements** if their truth tables are the same.

Example: For any statement A , A and $\neg(\neg A)$ are equivalent.

A	A	$\neg(\neg A)$
T	T	T
F	F	F

Example: Let's compare $\neg(A \wedge B)$ and $(\neg A) \vee (\neg B)$...

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$(\neg A) \vee (\neg B)$
T	T					
T	F					
F	T					
F	F					

A statement is a **tautology** if if the truth table outputs True for all inputs.

Example: "They'll win or they won't" is a tautology.

Example: The statement $A \vee \neg A$ is a tautology.

A	$\neg A$	$A \vee (\neg A)$
T	F	T
F	T	T

A statement is a **contradiction** if if the truth table outputs False for all inputs.

Example: "They'll win and they'll lose" is a contradiction.

Example: The statement $A \wedge \neg A$ is a contradiction.

A	$\neg A$	$A \wedge (\neg A)$
T	F	F
F	T	F

You try:

Construct truth tables for

(i) $(\neg A) \vee B$,

(ii) $\neg(A \vee B)$,

(iii) $(\neg A) \wedge (\neg B)$,

(iv) $(A \wedge B) \vee (\neg A \vee \neg B)$, and

(v) $(A \wedge B) \wedge (\neg A \vee \neg B)$.

Be sure to show all your steps by including columns for things like $\neg A$, $\neg B$, $A \vee B$, $A \wedge B$, $\neg A \vee \neg B$, and $\neg(A \vee B)$.

Implications

An **implication** is a statement of the form

“If statement A is true, then statement B is true.”

Notation: $A \Rightarrow B$ (L^AT_EX: \Rightarrow is `\Rightarrow`)

Examples

- (i) If Jill's enrolled in this class, then Jill's a student at CUNY.
- (ii) If Jill's a student at CUNY, then Jill's enrolled in this class.
- (iii) If $a < b$, then $-a > -b$.
- (iv) If $a < b$, then $a^2 < b^2$.
- (v) If it's raining outside, then 4 is even.
- (vi) If you're enrolled in this class, then 4 is odd.
- (vii) If 4 is odd, then 17 is even.

An implication “ $A \Rightarrow B$ ” is false when A is true and B is false, and is true otherwise.

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An implication “ $A \Rightarrow B$ ” is false when A is true and B is false, and is true otherwise.

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Compare this truth table to the one you computed for $(\neg A) \vee B$!

Implications

An **implication** is a statement of the form

“If statement A is true, then statement B is true.”

In this implication, A is called the **assumption** and B is called the **conclusion**.

Sometimes, in statements, the implication is hidden.

Example: The statement

“The square of a real number is non-negative”

can be restated as

“If x is a real number, then x^2 is non-negative.”

Recall: In Polya’s four-step plan, part of “understanding the problem” is restating the problem. Pulling apart an implication into explicit assumption and conclusion can help you prove the implication.

Other ways of saying “If A then B ”:

“ B if A ” and “ A only if B ”.