Polya's four-step plan for problem solving

1. Understand the problem.
2. Devise a plan.
3. Execute the plan.
4. Look back.

Understand the problem.

- What definitions, notation, etc. do you need to know before you start?
- Do some work, and make a guess.
  - Is there a smaller, similar problem that might give you some intuition?
  - Are there special cases that are easier to understand?
  - Can you do concrete examples?
  - Can you draw a picture?
- Which part of the problem statement is the assumption(s)? Which part is the conclusion(s)? What do you already know about these?
  (Note: Houston calls assumptions "hypotheses", which is not usually how we use that word. *Hypothesis* usually means a guess based on evidence.)
- Work backward and forward.
- Rewrite the problem in an equivalent way.

Devise a plan.

- Break into pieces.
  Step 1, step 2, step 3, . . .
  Case 1, case 2, case 3, . . .
- Find the right level.
  Does this problem need a big powerful theorem? Or can you do it just for the definitions?
- Give things names, so that you can refer to them or use them more easily.
  Let A be. . .
- Systematically choose a method.
  What standard methods have you already learned? Might any apply? Try one at a time.

Execute the plan.

Do that stuff you planned, checking and reassessing the plan at each step.

Look back.

- Sanity check!
  Examples, easy properties, etc.
  Can you find an example contradicting your solution?
- Revise, revise, revise.
  Did you use all your assumptions? Did you do things in the best order? Is there an easier/clearer path to your answer?
- Reflect.

You try: In your groups, for each of the following exercises:

(i) In the problem statement, what are the assumptions? What are the conclusions?
(ii) Do a couple of small examples, or do a similar smaller/easier version of the problem.
(iii) Solve the problem.

(a) Show that

$$\frac{a+b}{2} \geq \sqrt{ab} \qquad \text{for all} \quad 0 < a \leq b.$$

(b) Show that

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

for all positive integers a, b, and c.

(c) Let $f(x) = 1/(1-x)$. Define the function f^r by

$$f^r(x) := \underbrace{f(f(\cdots(f(f}_{r \text{ times}}(x)))\cdots)).$$

Find $f^{653}(56)$.

(d) Without using a calculator, show that $\sqrt[7]{7!} < \sqrt[8]{8!}$.

(e) Without using a calculator, show that $\sqrt{100001} - \sqrt{100000} < \frac{1}{2\sqrt{100001}}$.

(f) Bottle A contains a liter of milk and bottle B contains a liter of coffee. A spoonful of coffee from B is poured into A, and the contents are mixed well. Liquid from A is then poured into B until B has one liter of liquid. Is the fraction of coffee in A greater than the fraction of milk in B, or vice versa?
[Hint: Giving things names will be super important here! You may also assume that your spoon isn't giant.]