Welcome to Math 308!

Course info

Me: Professor Daugherty, zdaugherty@gmail.com
Website:
https://zdaugherty.ccnysites.cuny.edu/teaching/m308f18/
Textbooks:
How to Think Like a Mathematician, Kevin Houston
Intro to Mathematical Structures and Proofs, Larry J. Gerstein
Elementary Analysis: The Theory of Calculus, Kenneth A. Ross

Homework: due on Tuesdays in class, Posted on course website.
FINAL DRAFTS.
Exams: Midterms 10/16&18 and 12/6&11.
Portfolio: Final version due 12/18.

Homework 0: Before class on Tuesday 9/4, send me an email at zdaugherty@gmail.com with subject line "Math 308: Homework 0", answering the questions outlined on the website.

Course expectations

- Read posted sections before class, and bring your own copy of daily notes if needed (posted night before class).
- Come to class, participate, ask questions, work (possibly together) on in-class exercises.
- Come to office hours at least once in the semester. If you can't make my office hour, make an appointment.
- Out of class studying and work should be about 2-3 times the amount of time spent in class (5.5–7 hours/week). Find classmates to study and work with!
- Hand in "final draft" homework, typed up in LaTeX, on time. Get good practice with writing; using words and complete sentences. Ok to work with other people, but write-ups must be your own.
- If there are accessibility accommodations or exam conflicts to be organized, contact me as soon as possible.
- If you send me email, use complete sentences and be specific (ok to send pics of work!).

Definition

A set is a well-defined collection of objects. The objects in the set are called the elements or members of the set.
(Contrast: a list is an ordered collection of objects)
If x is an element of X, we write $x \in X$.

Examples:

(i) Set containing the numbers 1, 2, and 3 is $\{1, 2, 3\} = \{1, 3, 2\} = \{3, 2, 1\}$. The number 3 is an element of the set, i.e. $3 \in \{1, 2, 3\}$, but $6 \notin \{1, 2, 3\}$.

(ii) The set $\{1, 5, 12, \{a, b\}, \{5, 72\}\}$ is the set containing the numbers $1, 5, 12$, and the sets $\{a, b\}$ and $\{5, 72\}$.
Essentially: sets can contain sets as elements.

If the set X has a finite number of elements, then we say X is a finite set, in which case the number of elements is called the cardinality or size of X, denoted $|X|$.

Ex: The set $\{1, 2, a, b\}$ has cardinality 4;
the set $\{1, \{2, a, b\}\}$ has cardinality 2.

Some special sets:
$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, the integers.
$\mathbb{N} = \mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$, the natural numbers.
Notation:

$$\Big\{ \underbrace{\qquad\quad}_{\text{objects}} \Big| \underbrace{\qquad\quad}_{\text{conditions}} \Big\}.$$

Read $|$ as "such that" or "that satisfy".
For example,

$$\mathbb{Z}_{>0} = \{x \in \mathbb{Z} \mid x > 0\}.$$

More special sets:
Non-negative integers: $\mathbb{Z}_{\geqslant 0} = \{0, 1, 2, 3, \dots\} = \{x \in \mathbb{Z} \mid x \geqslant 0\}$.
Rational numbers: $\mathbb{Q} = \{p/q \mid p, q \in \mathbb{Z}, q \neq 0\}$.
Real numbers: $\mathbb{R}$. Tricky to define, but think: all decimal expansions. Ex: $0, 1, 1/3, \pi, -\sqrt{2}, \dots$
Irrational numbers: $\mathbb{R} - \mathbb{Q}$. For example: $\sqrt{2}$.
Complex numbers: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$, where $i = \sqrt{-1}$.
The empty set: $\varnothing = \{\}$ (nothing is in here)

Suppose X is a set. A set Y is a subset of X if every element of Y is an element of X, written $Y \subseteq X$. This is the same as saying

if $x \in Y$, then $x \in X$.

If $Y \subseteq X$ but X has at least one element that's not in Y, we say Y is a proper subset of X, written $Y \subsetneq X$ for emphasis.

Examples:

(i) The set $Y = \{1, \{3, 4\}, a\}$ is proper a subset of $X = \{1, 2, a, \{3, 4\}, b\}$.

(ii) The set of natural numbers is a proper subset of $\mathbb{Z}$. (Ignore Ex. 1.12(ii) - even numbers can be negative.)

(iii) The set $\{1, 2, 3\}$ is not a subset of $\{2, 3, 4\}$ or $\{2, 3\}$.

(iv) For any set X, we have $X \subseteq X$ and $\varnothing \subseteq X$.

Elements versus subsets: If $x \in X$, then $\{x\} \subseteq X$, and vice versa.

Example: Consider the set $X = \{x, \{x\}\}$. Then $x \in X$ and $\{x\} \subsetneq X$, but also we have $\{x\} \in X$.

Operations on sets

Let X and Y be sets.
The union of X and Y is

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\},$$

the set consisting of elements that are in X or in Y, or in both.

The intersection of X and Y is

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\},$$

consisting of elements that are in X and in Y.

The difference of X and Y, denoted $X \backslash Y$ or $X - Y$, is the set of elements that are in X but not in Y. Note: we do not require that Y is a subset of X. If Y <u>is</u> a subset of X, then $X - Y$ the complement of Y in X, denoted by Y^c.

The product of X and Y is

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\},$$

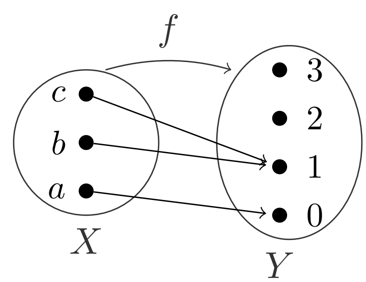
the set of all possible (ordered) pairs (x, y) where $x \in X$ and $y \in Y$.

Functions

Let X and Y be sets.

A function or map f from X to Y, written $f : X \to Y$, is an assignment of one $y \in Y$ for each $x \in X$. The unique element in Y associated to x is denoted $f(x)$. The set X is called the source or domain of f, and Y is called the target or codomain of f.

To describe a function f, we can use a formula, like $f(x) = x^2$. Or we can use a picture, like



Note: every element of X gets one element in Y, but not necessarily vice versa; and two distinct elements of X may map to the same element in Y.

Some examples:

(i) Fix some $c \in \mathbb{R}$. The constant function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = c$ only has the value c.

(ii) The cardinality of a set is a function given by

$$|| : \text{Finite sets} \to \mathbb{Z}_{\geqslant 0}.$$

(iii) The identity map on X is the map

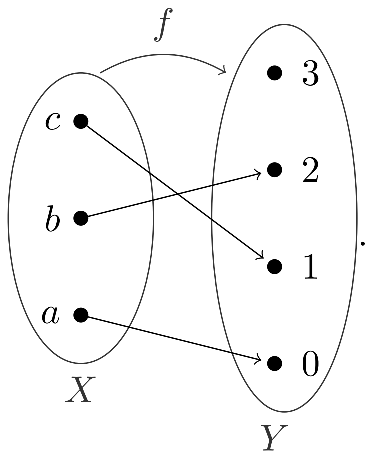
$$\mathrm{id} : X \to X \quad \text{given by} \quad f(x) = x \text{ for all } x \in X.$$

Non-example: The formula $f(x) = 1/(1 - x)$ does not define a function from $\mathbb{R}$ to $\mathbb{R}$, since it is not defined at $x = 1$.

A function $f : X \to Y$ is called one-to-one or injective if every element in Y gets mapped to by at most one $x \in X$.
Some examples of injective functions:

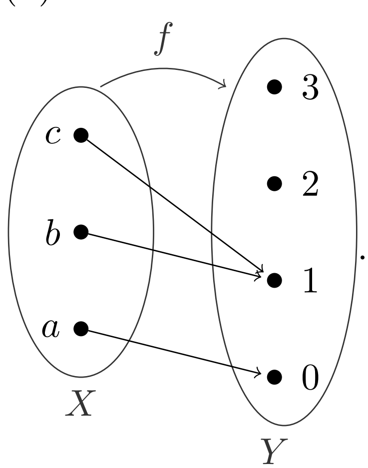
$$f(x) = 3x - 5 \text{ with domain } \mathbb{C},$$

$$f(x) = x^2 \text{ with domain } \mathbb{R}_{\geqslant 0},$$



A function $f : X \to Y$ is called one-to-one or injective if every element in Y gets mapped to by at most one $x \in X$.
Some examples of functions that are **not** injective:

$$f(x) = x^2 \text{ with domain } \mathbb{R},$$



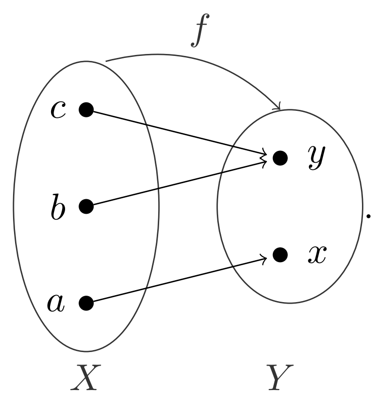
The image of a function $f : X \to Y$ is

$$f(X) = \{y \in Y \mid f(x) = y \text{ for some } x \in X\}.$$

A function is called onto or surjective if the codomain and the image are the same set.

Some examples of surjective functions:

$$f(x) = 3x - 5 \text{ with domain and codomain } \mathbb{R},$$
$$f(x) = x^2 \text{ with domain } \mathbb{R} \text{ and codomain } \mathbb{R}_{\geqslant 0},$$



A function is called onto or surjective if the codomain and the image are the same set.

Some examples of functions that are **not** surjective:

$$f(x) = 3x - 5 \text{ with domain } \mathbb{R} \text{ and codomain } \mathbb{C},$$
$$f(x) = x^2 \text{ with domain and codomain } \mathbb{R},$$