CHAPTER **19**

A study of Pythagoras' Theorem

Reason is immortal, all else mortal.
Pythagoras, *Diogenes Laertius (Lives of Eminent Philosophers)*

Pythagoras' Theorem is probably the best-known mathematical theorem. Even most non-mathematicians have some vague idea that it involves triangles and squaring something known as the hypotenuse.

Because the ideas in the previous chapters on 'How to read a theorem' and 'How to read a proof' are so important we will apply them to this famous theorem to see them in action. So in this chapter we will pull apart the theorem and its proof, we'll see a converse for it and also a generalization.

Statement of Pythagoras' Theorem

As you are a budding mathematician, you probably have a better idea than a non-mathematician of what the statement is, but here it is again.

Theorem 19.1

For a right-angled triangle the square of the hypotenuse is equal to the sum of the squares of the other sides.

Exercise 19.2

Use the ideas from Chapter 16, How to read a theorem, to analyse the theorem. Compare your analysis with the one given below.

Study of the theorem

We now analyse the theorem as though we were meeting it for the first time. Obviously we would check what all the words mean, for example, what is a hypotenuse? This is fairly obvious, but what about the other techniques in Chapter 16? We shall apply them now.

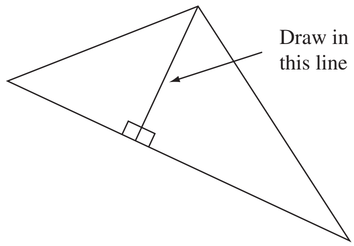

Figure 19.1 Any triangle gives two right-angled triangles

Find the assumptions and conclusions

The statement given for Pythagoras' Theorem is a good example of a statement not in the form '$A \Longrightarrow B$' or 'if..., then...'. We can rewrite it in this form in a number of ways. For example,

'If T is a right-angled triangle with sides a, b and hypotenuse c, then $c^2 = a^2 + b^2$.'

This make it obvious that the assumptions concern all right-angled triangles: 'T is a right-angled triangle with sides a, b and hypotenuse c' and that the conclusion is an equation relating the lengths: '$c^2 = a^2 + b^2$'.

Rate the strength of the assumptions and conclusions

Let's rate the assumptions and conclusions. The assumption is about right-angled triangles. Certainly, there are many examples of these, but they are only a small subset of *all* triangles. Thus we may be tempted to say that this is quite weak but not too weak. But consider this. For any triangle we can produce two right-angled triangles; see Figure 19.1. Thus, despite initial impressions, this theorem will tell us something about all triangles. This means it is a very weak assumption. That is good. The more examples a theorem applies to the better.

The conclusion is a very concrete and definite statement. It is an equation. If we just had an inequality (say $c^2 \geq a^2 + b^2$), then the conclusion would be less impressive. This is a strong conclusion.

Note also that the conclusion allows us to calculate: given the lengths of any two sides of a right-angled triangle we can calculate the third. Being able to calculate is always good for a conclusion.

From this we know we have a great theorem which will be very useful as it allows us to calculate attributes of a large number of objects. In fact, as you may know, it allows us to calculate distances in Cartesian coordinates.

Compare with previous theorems

We don't have many theorems to compare with as this is a very simple theorem from the foundations of geometry. Another theorem from the foundations is that in a triangle (not necessarily right-angled) the angles add up to 180 degrees (or 2π radians if you like).

We can see this is similar in the sense that given two angles we can calculate the third. It is different in the sense that it is about angles rather than about lengths. Thus they are quite complementary!

Now that we have observed a theorem that deals with lengths and one that deals with angles we can ask whether there exists one that links both. Indeed there is one.

Exercise 19.3

Find a theorem that relates angles and lengths in a triangle.

Observe the detail

There are not many details to observe in Pythagoras' Theorem as it is rather simple. The detail to remember perhaps is that the triangle is right-angled – not, for example, isosceles. (See the exercises at the end of the chapter for some infamous examples of mistakes with the statement.)

Classify what the theorem does and how it can be used

We have already discussed above what the theorem does: it allows us to calculate the length of a side of a triangle given the lengths of the other two.

Draw a picture

Since this is a geometric theorem – it is about triangles – then we have good reason to draw a picture. In this case draw lots of triangles and measure the lengths. Do the lengths satisfy the equation? They should do but remember we can only measure lengths approximately.

The classic picture is the (3, 4, 5)-triangle given in Figure 19.2. This is often mistakenly given as a proof of the theorem. However, it is only a single example, not a proof.

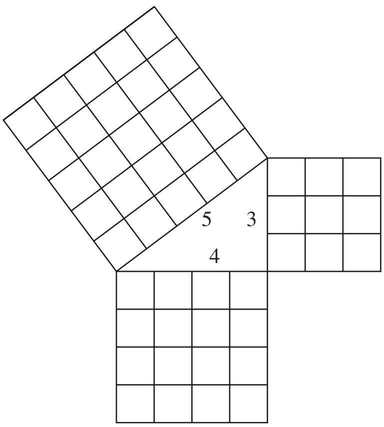

Figure 19.2 The classic (3, 4, 5)-triangle

Apply the theorem to simple examples

In drawing the pictures we have already applied the theorem to simple examples such as the (3, 4, 5)-triangle.

Apply the theorem to trivial and extreme examples

The assumptions concern right-angled triangles. What are trivial and extreme examples of such objects? I would say that the trivial examples are when $a = 1$ and $b = 1$ so $c = \sqrt{2}$. This is a triangle with two sides of rational lengths and one side of irrational length. One could view this as a counterexample to the statement 'If two sides of a triangle are rational, then so is the third.'

Another useful example is $a = 1$, $b = 2$, and so $c = \sqrt{5}$. This just shows us what happens when one side is twice the length of the other. Another interesting example is $a = 1$, $b = \sqrt{2}$, and so $c = \sqrt{3}$. I like this, it involves the first three numbers 1, 2 and 3. The triangle $a = 1$, $b = \sqrt{3}$ and $c = 4$ is good as it has an angle of $\pi/6$.

Now for extreme examples. What is an extreme right-angled triangle? We can view $a = b$ as an extreme triangle because it is a very special case of a right-angled triangle. Here $c^2 = 2a^2$, so $c = \sqrt{2}a$. This means that if a is rational, then c cannot be rational (since a rational number times an irrational number, in this case $\sqrt{2}$, is irrational).

Another extreme we can go to is to make one of the sides very big or very small compared with the other. If we let b tend to zero, then c^2 must tend to a^2 as b^2 gets small. In other words c tends to a. We can see this in a diagram. Just draw a triangle with a very small b as compared with a; you can see that a and c are almost equal. Thus we have shown that the algebra and the geometry are linked in the way we would expect. The algebra agrees with the geometry.

Note that if b equals zero, then we do not have a triangle!

Apply the theorem to non-examples

Let us consider non-examples of the theorem. Here non-examples are triangles that are not right-angled. However, in this case how do we even define hypotenuse? This is always defined as the length opposite the right-angle. There is no way of identifying a special length, so there is no way to identify which should go on the right-hand side of $c^2 = a^2+b^2$. We need another method of identifying c. Maybe we have to take the longest side. This is reasonable since, for $c^2 = a^2 + b^2$ to be true, we must have $c > a$ and $c > b$. To see this, consider that $b^2 > 0$ is true, thus $c^2 > a^2$, and so $c > a$. Similarly $c > b$.

If you try drawing a few examples, then you will probably see that the equation does not hold, even taking into account approximations in measuring. This should lead us to ask 'Are there *any* non-right-angled triangles for which $c^2 = a^2 + b^2$?' We shall deal with this in the next section when we ask 'Is the converse true?'

We shall also look at non-examples in Exercises 19.12.

Rewrite in symbols or words

We have already written the theorem in as many symbols as possible when we gave the 'if …, then …' statement. Note that is not sufficient to say only '$c^2 = a^2 + b^2$'. We have to explain what a, b and c are, and that c is the hypotenuse in a right-angled triangle.

Proof of Pythagoras' Theorem

As mentioned earlier, it is not enough to show the theorem works approximately for some examples or even perfectly in some special cases such as the (3, 4, 5)-triangle. We need to prove that it holds for all triangles. Maybe you were told that the theorem was true by some authority figure in the past and that is good enough for you. However, a central aim of this book is to encourage you to think for yourself and that involves checking any argument given to you.

We will now see a proof of the theorem. There are literally hundreds of proofs.[1] My favourite proof is geometrical.

Proof (of Pythagoras' Theorem). The proof can be shown using the two squares in Figure 19.3. To draw the first square begin by drawing a general triangle with sides a and b and then extend these edges by lengths b and a respectively. Then we can complete the drawing to get the square on the left-hand side of Figure 19.3.

We can draw another square like the one on the right-hand side of the figure. From the figure we can see that both squares have equal area and so we can conclude that

$$\begin{aligned}\text{Area of left square} &= \text{Area of right square}\\ c^2 + (4 \times \text{Area of } (a, b)\text{-triangle}) &= a^2 + b^2 + (4 \times \text{Area of } (a, b)\text{-triangle})\\ c^2 &= a^2 + b^2.\end{aligned}$$

□

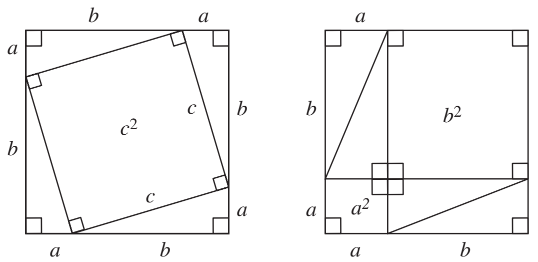

Figure 19.3 Proof of Pythagoras' Theorem

[1] There are over 360 in Elisha Loomis, *The Pythagorean Proposition*, National Council of Teachers of Mathematics, 1968.

Exercise 19.4

Use the ideas from Chapter 18, How to read a proof, to analyse the proof. Compare your analysis with the one given below.

Not all the suggestions in Chapter 18 are relevant. Let us try some of them to help pull the proof apart.

Find where the assumptions are used

Where have we used the assumption that the triangle is right-angled? It is used in constructing the first square. If sides a and b did not meet at a right-angle, then we could not construct a square. And hence could not conclude that the area was the same as the square on the right.

Check the text

If you check the text, then you may see that one fact has been used but not explicitly stated. The picture is very convincing in that the area labelled c^2 in the left-hand picture certainly looks square. But notice that we have not proved that it is square. We know that all the edges are the same length but this does not mean that the shape is a square – think of a diamond or kite shape.

How do we know it is truly a square? Well, we need to show that one of the internal angles is right-angled, once we get it for one, the same proof will work for all. (Can you see why?) Call the internal angle γ.

Suppose that the triangle in Figure 19.4 has small angle α and larger angle β. Then we can see that $\alpha + \beta + \gamma = 180°$ since we have a straight line:

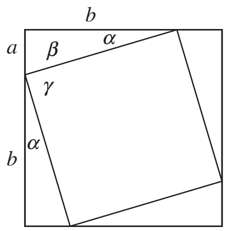

Figure 19.4

But we know that $\alpha + \beta + 90° = 180°$ since the angles in a triangle add up to 180° and our triangle is right-angled. Using these two equations we deduce that γ is 90° as required.

If we add this explanation, then the argument will be even more convincing. True, the picture looks convincing already, but we have to be careful with pictures – they can mislead.

Exercise 19.5

How else could the picture fool us?

What about the converse?

Let's now look at an important technique for exploring theorems: is the converse true? To do this it helps to write the statement in an 'If …, then …' form:

> If T is a right-angled triangle with sides a, b and hypotenuse c, then $c^2 = a^2 + b^2$.

The converse will be

> If $c^2 = a^2 + b^2$, then T is a right-angled triangle with sides a, b and hypotenuse c.

Note that this does not quite make sense because in the assumptions we do not know what a, b and c are. Let's rewrite Pythagoras' Theorem as the following:

> Let T be a triangle with sides of length a, b and c with c the longest. If T is a right-angled triangle, then $c^2 = a^2 + b^2$.

Now the converse becomes:

Theorem 19.6 (Converse of Pythagoras' Theorem)

Let T be a triangle with sides of length a, b and c with c the longest. If $c^2 = a^2 + b^2$, then T is a right-angled triangle.

Is this true or not? It is! But why?

Proof. Let T be a triangle with sides of length a, b and c where C is the angle opposite to the side of length c, then the Cosine Rule states that

$$c^2 = a^2 + b^2 + 2ab\cos C.$$

Now suppose that $c^2 = a^2 + b^2$, then we have $2ab\cos C = 0$. As a and b cannot be zero we must have $\cos C = 0$. This implies that $C = 90° + 180°n$, where n is some integer. Since $0 < C < 180°$ we must have $C = 90°$, i.e. T is a right-angled triangle. □

Exercise 19.7

Apply the methods of Chapter 18, How to read a proof, to the above proof.

Remark 19.8

The proof of the converse used the Cosine Rule, the proof of which uses Pythagoras' Theorem (see page 22 and following). Note that the Cosine Rule is more general than Pythagoras' Theorem, since we can deduce Pythagoras from it: if we have a right-angled triangle, then $C = 90°$ and so $c^2 = a^2 + b^2 - 2ab \times 0 = a^2 + b^2$.

Thus, we can see that a more general statement can use the simpler theorem in its proof.

Since Pythagoras' Theorem and its converse are true we have two equivalent statements (i.e. If $A \implies B$ and $B \implies A$, then $A \iff B$). This means we can state the following theorem.

Theorem 19.9

Let T be a triangle with sides of length a, b and c with c the longest. Then T is a right-angled triangle if and only if $c^2 = a^2 + b^2$.

Proof. We can prove this by combining the proof of Pythagoras we gave and then using the converse argument we just gave. Note that we can't just use the Cosine Rule in both directions since Pythagoras' Theorem is used in the proof of the Cosine Rule! Did you spot that? □

A bit more on understanding a converse

Suppose that we have a triangle with sides of length 2.0, 2.1 and 2.9. We have $2.0^2 + 2.1^2 = 2.9^2$. The triangle is therefore right-angled.

Exercise 19.10

Did we use Pythagoras' Theorem or its converse to deduce this?

Suppose that we have a triangle of sides of length 3.6, 7.7 and 8.4. In this case, $3.6^2+7.7^2 = 72.25 \neq 70.56 = 8.4^2$. Thus the triangle is not right angled.

Exercise 19.11

Did we use Pythagoras' Theorem or its converse to deduce this?

A common mistake in answering this is to make an argument that the sides do not satisfy the equation $c^2 = a^2 + b^2$ so this can't use Pythagoras' Theorem. Therefore, it must use the converse. This is wrong.

The correct argument is that Pythagoras' Theorem says that if you have a right-angled triangle, then it sides satisfy the equation. Thus if the equation is *not* satisfied, then there is no way that the triangle could be right-angled. In effect, we are using the contrapositive statement to Pythagoras' Theorem, i.e. an equivalent statement: If $c^2 \neq a^2 + b^2$, then T is not right-angled.

Exercises

Exercises 19.12

(i) Pythagoras' Theorem is often misquoted. In the film *The Wizard of Oz*, the Scarecrow is given a diploma and to show how clever he has become he points his finger to his temple and says 'The square root of the hypotenuse is equal to the sum of the square roots of the other sides for an isosceles triangle.'

This is most definitely not Pythagoras' Theorem as it involves square roots and isosceles triangles. However, just because it is not Pythagoras' Theorem does not mean that the statement is false! Apply our methods to analyse this statement. Is it true? If not, then give a counterexample.

(ii) The theorem is also misquoted in the long-running animated comedy *The Simpsons*. In the episode *\$pringfield (or, how I learned to stop worrying and love legalized gambling)* Homer finds some glasses and puts them on and à la Scarecrow puts his finger to his temple and states 'The sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side.' Another character then shouts 'That's a right-angled triangle, you idiot!'

Why are Homer and the other character *both* wrong?

(iii) Many theorems from lower-level mathematics are given without proof. Find as many examples of this as you can. Try to find proofs for them from other sources and analyse these using the methods of Chapter 18, How to read a proof. Some examples you may like to try:
  (a) The sum of angles in a triangle is 180 degrees.
  (b) Subtracting a negative is equivalent to adding a positive.
  (c) The value of π is $3.14159\ldots$
  (d) The definition of sine and cosine don't depend on the triangle. (You will need to know about similar triangles.)
  (e) For any angle θ, $\sin^2\theta + \cos^2\theta = 1$.
  (f) The area of a circle of radius r is πr^2.

(iv) Consider the triangles with sides of the following lengths. Decide which are right-angled and state whether you used Pythagoras' Theorem or its converse.
  (a) 7, 24, 25, (b) 28, 45, 52,
  (c) 36.9, 80.0, 88.0, (d) 0.8, 1, 4, 1.7.

(v) Construct a right-angled triangle such that the hypotenuse has irrational length but the other two sides have rational length.

(vi) Construct a right-angled triangle such that the hypotenuse has rational length but the other two sides have irrational length.

(vii) Let X and Y be distinct points in a plane. Draw a line between them. At the midpoint draw a line perpendicular to the line. See Figure 19.5. This line is called the **perpendicular bisector** of X and Y.

Show that for every point p on the perpendicular bisector the distance from X to p and the distance from Y to p are the same. We say p is **equidistant** from X and Y.

(viii) Go back to previous chapters and exercises and re-analyse the theorems and proofs. Did you observe facts that you did not observe before?

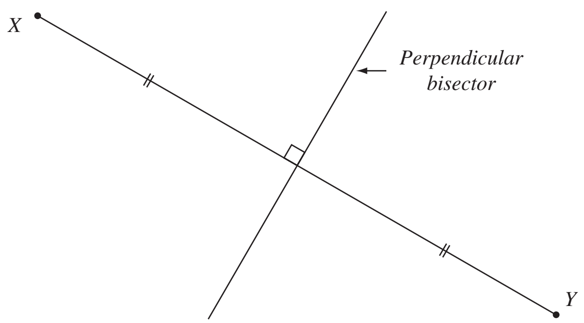

Figure 19.5 Perpendicular bisector

Summary

- Pythagoras' Theorem is: Let T be a triangle with sides of length a, b and c with c the longest. If T is a right-angled triangle, then $c^2 = a^2 + b^2$.
- The converse of Pythagoras' Theorem is: Let T be a triangle with sides of length a, b and c with c the longest. If $c^2 = a^2 + b^2$, then T is a right-angled triangle.