HOMEWORK 8
MATH 308
DUE: 11/6/2018

1. Show the following using proof by cases.
   (a) The square of any integer is of the form $3k$ or $3k+1$ for some $k \in \mathbb{Z}$.
   (b) For all $x, y \in \mathbb{R}$, we have
   $$|xy| = |x||y| \qquad \text{and} \qquad \big||x| - |y|\big| \leq |x - y|.$$
   (c) For sets A and B, we have $A \cup B = (A \cap B) \cup (A - B) \cup (B - A)$.
2. Prove the following by induction.
   (a) We have $n^3 + 2n$ is a multiple of 3 for all $n \in \mathbb{Z}_{\geq 0}$.
   (b) We have $\sum_{i=1}^{n} 2i - 1 = n^2$ for all $n \in \mathbb{Z}_{\geq 0}$.
   (c) Suppose $A_1, A_2, \ldots A_N$ and $B_1, B_2, \ldots, B_N$ are sets such that
   $$A_i \subseteq B_i \qquad \text{for all } 1 \leq i \leq N.$$
   Then
   $$\bigcup_{i=1}^{N} A_i \subseteq \bigcup_{i=1}^{N} B_i.$$
   [Hint: With some problems, you may have to do more than one base case; and you may have to prove a lemma for one of those. For example, here, you can check that the claim holds trivially for $N = 1$. But you will also need to prove the claim for $N = 2$ before you can do the inductive step, using a direct proof. See the .tex file.]
3. The following is an example of why checking the base case is important.
   Let $P(n)$ be the statement "$2^n < 2^{n-1}$".
   (a) Show that $P(n) \Rightarrow P(n+1)$. Namely, assume $2^n < 2^{n-1}$ for some $n \geq 1$, and use that to show that $2^{n+1} < 2^n$.
   (b) Check the base case, $P(1)$.
   (c) Prove that $P(n)$ is actually false for all $n \in \mathbb{Z}_{\geq 1}$.
4. Each of the following claims are false, but are accompanied by subtly flawed proofs by induction. For each, find the flaw.
   (a) **Claim:** For every positive integer n, if x and y are positive integers with $\max(x, y) = n$, then $x = y$.[1]

   Proof by induction. First, for $n = 1$, if $\max(x, y) = 1$ and x and y are positive integers, then we have $x = 1$ and $y = 1$.
   Next, fix $n \geq 1$ and assume that for any positive integers x', y' such that $\max(x', y') = n$, then $x' = y'$. Let x and y be positive integers such that $\max(x, y) = n + 1$. Then $\max(x - 1, y - 1) = n$, so by the inductive hypothesis, $x - 1 = y - 1$. It follows that $x = y$. Thus, by induction, the claim holds for all $n \geq 1$. □

   (b) **Claim:** If $\mathcal{H}$ is a set of n horses, then all the horses in $\mathcal{H}$ are the same color.

[1] In other words, if x and y are (possibly equal) positive integers, and the biggest of them is equal to n, then $x = y$.

Proof by induction. If the set only has one horse in it, then, trivially, all the horses in that set have the same color. Now assume that the claim holds for a fixed $n \geq 1$, so that all the horses in any set of n horses are the same color. Now consider a set $\mathcal{H}$ of $n+1$ horses. First, line up those horses in order: the first n of these horses all must be the same color, and the last n of these must also be the same color. Because the set of the first n horses and the set of the last n horses overlap, all $n+1$ must be the same color. Thus, by induction, all the horses in any finite set of horses must have the same color. $\square$