HOMEWORK 7–UPDATED VERSION
MATH 308
DUE: 10/30/2018

1. Axioms.
   (a) Look up and state the Zermelo-Fraenkel set theory axioms, and find something of importance said about them.
   (b) Look up and state Euclid's Axioms, giving both the words and a symbolic translation of them. What's the difference between Euclid's Axioms and Euclid's Postulates?
   (c) Find one more historically important set of mathematical axioms (you can tell they're probably important if they's named after a well-known mathematician).

2. Theorems versus Propositions versus Lemmas: Go to https://arxiv.org/archive/math and pick a topic. Skim three recent papers that are 30 pages or longer (that are not self-proclaimed "survey articles"), accounting only for how many theorems there are, how many propositions there are, and how many lemmas they are; as well as which ones are discussed in the introduction and how (spend no more than 5 minutes per paper). Give your numerical data, and a commentary on anything you noticed. Include a references section in your homework (see HW2 for guidance).

3. For each of the following definitions, find examples and non-examples. Find at least one trivial example, one standard example, one extreme example, and one non-example.
   (a) A continuous function $f : \mathbb{R} \to \mathbb{R}$ is called *convex* if for all $x < y$ and $0 \leq t \leq 1$, we have
   $$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$
   [Hint: Informally, a convex function is a continuous function whose value at the midpoint of every interval in its domain does not exceed the arithmetic mean of its values at the ends of the interval. In particular, think about what you've been told "convex" means before, and try to figure out why this definition matches that intuition.]
   (b) Let x be a natural number. Take the sum of the squares of the digits to produce a new integer. Repeat this process until it produces a 1 or repeats indefinitely. The integer x is called a *happy number* if the process ends with a 1. (Non-standard definition)
   (c) Let A be a set, and let $\star : A \times A \to A$ be a binary operation. We call the pair $(A, \star)$ a *group* if
      (i) $\star$ is associative;
      (ii) A has an identity element e (with respect to $\star$); and
      (iii) for every $a \in A$ there exists some $b \in A$ (called the *inverse* of a) such that $a \star b = e$ and $b \star a = e$.

4. Investigate the following theorem statements, following the steps in Chapter 17, wherever applicable.
   (a) If A and B are finite sets, then $|A \times B| = |A||B|$.
   (b) Let n be an integer with $n \geq 3$. For n distinct points on a circle, connect each pair of consecutive points by a straight line. The sum of the interior angles of the resulting shape is $(n-2) \times \pi$.

5. Use the techniques of Chapter 18 to analyze the following proof that, for sets A and B we have $A \neq B$ if and only if $(A - B) \cup (B - A) \neq \emptyset$. Don't forget to consider the extreme case of empty sets!

   Proof. Suppose that $A = B$. Then

   $$(A - B) \cup (B - A) = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset.$$

   If $A \neq B$, then there exists $x \in A$ but $x \notin B$ or there exists $x \in B$ but $x \notin A$. In the former we have $x \in A - B$, and hence $x \in (A - B) \cup (B - A)$. Therefore $(A - B) \cup (B - A) \neq \emptyset$. A similar reasoning proves the result in the latter possibility.

   □

6. Many theorems from lower-level mathematics are given without proof. Find proofs for at least two of the following from other sources, and analyze them using the methods of Chapter 18.
   (a) The sum of angles in a triangle is $180°$.
   (b) Subtracting a negative is equivalent to adding a positive.
   (c) Zero times any number is zero.
   (d) The value of π is $3.14159\ldots$.
   (e) For any angle θ, $\sin^2\theta + \cos^2\theta = 1$.
   (f) The area of a circle of radius r is πr^2.