YOUR NAME HERE
Math 308
Homework 11
12/11/2018

For this assignment only, if you prefer, you may turn in partially or fully hand-written solutions.

For problems 1–3, you may use our old working knowledge of number systems like $\mathbb{Z}$ and $\mathbb{R}$. For problems 4–6, use the new set-theoretic definitions we developed, and only use assumptions about their properties as listed in the prompt.

1. For the following relations on X determine whether they are reflexive, symmetric, and/or transitive. State whether they are equivalence relations or not and if they are describe their equivalence classes.
   (a) Let $X = \mathbb{Z}$ and define $\sim$ by $x \sim y$ if $x - y$ is odd.
   (b) Let $X = \mathbb{R}$ and define $\sim$ by $x \sim y$ if $xy \neq 0$.
   (c) Let $X = \mathbb{R} \times \mathbb{R}$ and define $\sim$ by $(a, b) \sim (c, d)$ if $(a - c)(b - d) = 0$.

2. Let $\sim$ be an equivalence relation on a set A, and for each $a \in A$, let

$$[a] = \{b \in A \mid a \sim b\}$$

   denote the *equivalence class* of x.
   (a) Prove that if $x \in [y]$, then $y \in [x]$.
   [See notes.]
   (b) Prove that for all $x, y \in A$ that either $[x] \cap [y] = \emptyset$ or $[x] = [y]$.
   [Use part (a)).]
   (c) Prove that the equivalence classes partition A (see notes).
   [Use part (b)).]

3. Let $\sim$ be the equivalence relation on $\mathbb{Z}$ given by $a \sim b$ if $a \equiv b \pmod{n}$. Define $\mathbb{Z}_n = \{[a] \mid a \in \mathbb{Z}\}$, where $[a]$ is the equivalence class of a modulo n.
   (a) Briefly explain why it makes sense to define $+ : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ and $\cdot : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ by

$$[a] + [b] = [a + b] \quad \text{and} \quad [a] \cdot [b] = [a + b].$$

   [If $[a] = [a']$ and $[b] = [b']$, do we know that $[a] + [b] = [a'] + [b']$?]
   (b) Briefly check that $\mathbb{Z}_n$ is a commutative ring.
   (By "briefly", I mean you can say things like "since addition and multiplication satisfy __ in $\mathbb{Z}$, we have the same properties in $\mathbb{Z}_n$.")
   (c) Find an n so that $\mathbb{Z}_n$ is not a field (justify your answer).

4. **Natural numbers.** For the following, use the set theoretic definition of $\mathbb{Z}_{\geq 0}$ developed in class. Let $a, b, c \in \mathbb{Z}_{\geq 0}$.

   Recall, we defined $\leq$ by
   $$a \leq b \quad \text{if} \quad b = S(S(\cdots S(a)\cdots)).$$
   and addition by
   $$a + 0 = a; \text{ and } a + S(b) = S(a+b),$$
   where $S : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is the successor function.

   Using these definitions of $+$ and $\leq$, prove that if $a \leq b$ then $a + c \leq b + c$.
   [You may want to use induction on c. You may assume that addition is associative.]

5. **Rational numbers.** For the following, use the set theoretic definition of $\mathbb{Q}$ developed in class. You may assume all properties we listed about addition, multiplication, and comparisons in $\mathbb{Z}$, like associativity, commutativity, additive inverses, etc.
   (a) Show that $\dfrac{0}{a} = \dfrac{0}{b}$ for all $a, b \in \mathbb{Z}_{\neq 0}$.
   (b) We defined addition by
   $$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d + b \cdot c}{b \cdot d} \qquad \text{and} \qquad \frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d},$$
   where the addition and multiplication on the right-hand-side of each equation is in $\mathbb{Z}$. We also said that for any $a \in \mathbb{Z}$, that in $\mathbb{Q}$, a is defined as $\frac{a}{1}$.
   Using these definitions, prove the following.
   (i) We have $\dfrac{a}{b} + 0 = \dfrac{a}{b}$ for all $\dfrac{a}{b} \in \mathbb{Q}$.
   (ii) We have $\dfrac{a}{b} + \left(\dfrac{-a}{b}\right) = 0$ for all $\dfrac{a}{b} \in \mathbb{Q}$.
   (iii) We have $\dfrac{a}{b} \cdot \dfrac{b}{a} = 1$ for all $\dfrac{a}{b} \in \mathbb{Q}$ with $b \neq 0$.
   [You'll need to show that $\frac{c}{c} = 1$ for all $c \in \mathbb{Z}_{\neq 0}$.]

6. **Real numbers.** Recall that the set $\mathcal{R}$ of Dedekind cuts is the subsets of $\mathbb{Q}$ satisfying properties **(i)**–**(iii)** on page 30 of "Elementary analysis" (see below), with operations defined set-theoretically. As in class, define $0^* = \{y \in \mathbb{Q} \mid y < 0\}$. You may assume all properties we listed about addition, multiplication, and comparisons in $\mathbb{Q}$, like associativity, commutativity, additive inverses, etc.

   For $\alpha, \beta \in \mathcal{R}$, we defined $\alpha + \beta = \{a + b \mid a \in \alpha, b \in \beta\}$.
   (a) Briefly verify that $0^* \in \mathcal{R}$.
   (b) Prove that for $\alpha, \beta \in \mathcal{R}$, we have $\alpha + \beta \in \mathcal{R}$.
   [You get to assume that α satisfies **(i)** $\alpha \neq \mathbb{Q}, \emptyset$; **(ii)** if $r \in \alpha$ and $s \in \mathbb{Q}$ satisfy $s < r$, then $s \in \alpha$; and **(iii)** α doesn't have a maximum (and similarly for β). Your goal is to show that $\alpha + \beta$ also satisfies those three properties.]
   (c) Briefly check that $\{a \in \mathbb{Q} \mid a^3 < 2\}$ is a Dedekind cut, but $\{a \in \mathbb{Q} \mid a^2 < 2\}$ is not.
   [You many use arithmetic properties of $\mathbb{Q}$ without proof.]
   (d) Show that $\alpha + 0^* = \alpha$ for all $\alpha \in \mathcal{R}$.