YOUR NAME HERE
Math 308
Homework 10
12/4/2018

Throughout, let n be a non-zero integer, and let a, b, and c be integers.

1. Find the least residue of $a \pmod{n}$ for each of the following:
   (a) $a = 16, n = 5$; (b) $a = 545, n = 12$; (c) $a = -33, n = 22$, (d) $a = 7, n = 7$.
2. Show that if $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$ for all $k \in \mathbb{Z}_{\geq 0}$.
3. Division.
   (a) Show that if $\gcd(a, b) = 1$, then $ac \equiv bc \pmod{n}$ implies $a \equiv b \pmod{n}$.
   (b) Prove or disprove: For all integers a and b, if $ab \equiv 0 \pmod{n}$, then $a \equiv 0 \pmod{n}$ or $b \equiv 0 \pmod{n}$.
   (c) Prove or disprove: For all integers $n > 3$ and $a, b \in \mathbb{Z}$, if $a^2 \equiv 4 \pmod{n}$, then $a \equiv 2 \pmod{n}$.
4. Rewrite, fine-tune, and complete the book's proof of Fermat's little theorem.
5. While it is not true that $x^m \equiv_m x$ for all $m \in \mathbb{Z}_{\geq 2}$, $x \in \mathbb{Z}$, some generalizations of Fermat's little theorem hold.
   (a) Find an example of $m \in \mathbb{Z}_{\geq 2}$ and $x \in \mathbb{Z}$ where $x^m \not\equiv_m x$.
   (b) Show that $x^3 \equiv_6 x$ for all $x \in \mathbb{Z}$.
6. Classify each of the following as finite, countably infinite, or uncountable. Justify/prove your answers.
   (a) The positive prime integers, $\{p \in \mathbb{Z}_{>0} \mid p \text{ is prime }\}$.
   (b) The rational numbers with denominators a power of 2, $\{n/2^m \mid n \in \mathbb{Z}, m \in \mathbb{Z}_{\geq 0}\}$.
   (c) The complex numbers, $\mathbb{C}$.
   (d) The irrational numbers, $\mathbb{R} - \mathbb{Q}$.
7. Show that for any set A, we have $|A| < |\mathcal{P}(A)|$, where $\mathcal{P}(A)$ is the power set of A. (See class notes.)