PROOF LAB II: INDUCTION

Instructions:

Step 1 (Today).
Work together with your group to outline the full set of exercises.

Step 2 (This weekend and next week).
Each person: pick 8 $\star$'s worth of exercises, and write up careful drafts of the proofs. Include key definitions and thorough reasoning—think about writing to convince a student taking Bridge next semester. Make sure a novice could follow and locate all critical components of the proof. **See portfolio guidelines for further instruction.**

Step 3 (11/8 $\pm$ 2 days).
Swap drafts with a peer editor and give/get feedback. (Get a peer's email address and sending them a pdf before class on Thursday.)

Provide written constructive criticism for your partner: Are key definitions included? Is the reasoning mathematically accurate? Are the key components of the proof easy to recognize? Is the notation clear? Would a few additional comments help a novice reader to follow more easily? Can the explanation be made more clean and concise?

Step 4 (Thursday 11/15).
Turn in your proof second drafts, and comments from at least one peer editor on first drafts. (You will provide two versions of each proof–one first draft with comments, and one rewritten draft. Homework without peer editor comments will not be accepted.)

Outlining your proof:

1. Define $P(n)$.
   Rewrite the claim in the best way for you to use induction.
2. Compute base case(s).
   Usually $P(0)$ or $P(1)$. You might have to do more than one base case to get the induction step to work.
3. Explicitly state your goal.
   "Assume $P(n)$ and prove $P(n+1)$, which is..."
4. Do inductive step.
   This is usually where the real work happens.
5. State your conclusion.

Rewrite your proof: Your final draft should read something like
"We will prove this claim by induction on n. First, for $n = a$,* we have [COMPUTATION]. Next, fix $n \geq a$ and assume [STATE THE CLAIM] for that value of n. Then [USE THAT ASSUMPTION TO SHOW CLAIM FOR $n+1$]**. Thus, by induction, the claim holds for all $n \geq a$."

*where a is something like 0 or 1 or 5—whatever the lower end of the domain of the problem is.
**Don't forget to point out where you use the "Induction Hypothesis".

Problems. Prove each of the following using proof by induction. Assume throughout that n is an integer.

I. ($\star$) If $x_1, x_2, ..., x_n$ are odd integers, then their product,

$$\prod_{i=1}^{n} x_i \qquad \text{is also odd.}$$

[Note that $x_1, x_2, ..., x_n$ is an arbitrary list of odd numbers (you don't get to choose what these numbers are!).]

II. ($\star$) For all odd natural numbers n, $n^2 - 1$ is divisible by 8.
[Warning: <u>Do not induct on n</u>. Instead, start by writing what it means for n to be odd.]

III. ($\star\star$) For all $n \geq 1$ and $0 \leq x \leq \pi$, we have $\sin(nx) \leq n \sin(x)$.
[You may use the angle addition formula for $\sin(x)$.]

IV. ($\star\star$) Let $a_n = 2^{2^n} + 1$. Then for all $n \geq 2$, the last digit of a_n is 7. †
[Hint: Rephrase "last digit is 7" in terms of remainders.]

V. ($\star\star\star$) The *Fibonacci Numbers* $F_1, F_2, \ldots,$ are defined by the recursive rule

$$F_1 = 1 \quad F_2 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \text{ for all } n \geq 3.$$

For all $n \geq 1$, F_{5n} is divisible by 5.
[Hint: For the induction step, let r be the remainder of F_{5n-1} when divided by 5.]

VI. ($\star\star\star$) If u and v are differentiable functions of x, then

$$\frac{d^n}{dx^n} uv = \sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k},$$

where $u_i = \frac{d^i}{dx^i} u$ and $v_i = \frac{d^i}{dx^i} v$.
[You will need to prove a lemma that shows $\binom{r}{s} + \binom{r}{s+1} = \binom{r+1}{s+1}$ for all $r, s \in \mathbb{Z}_{\geq 0}$, to be proven directly.]

†Fermat conjectured that for $n \geq 1$, a_n is always prime. Indeed, $a_1 = 5, a_2 = 17, a_3 = 257, a_4 = 65,537$ are all prime. However, the conjecture fails for many large values of n (Fermat had a hard time checking because a_i gets really large quickly).