PROOF LAB II: INDUCTION – HINTS AND TIPS.

Outlining your proof:

1. Define $P(n)$.
   Rewrite the claim in the best way for you to use induction.
2. Compute base case(s).
   Usually $P(0)$ or $P(1)$. You might have to do more than one base case to get the induction step to work.
3. Explicitly state your goal.
   "Assume $P(n)$ and prove $P(n+1)$, which is..."
4. Do inductive step.
   This is usually where the real work happens.
5. State your conclusion.

Rewrite your proof: Your final draft should read something like
"We will prove this claim by induction on n. First, for $n = a$,* we have [COMPUTATION]. Next, fix $n \geq a$ and assume [STATE THE CLAIM] for that value of n. Then [USE THAT ASSUMPTION TO SHOW CLAIM FOR $n + 1$]**. Thus, by induction, the claim holds for all $n \geq a$."

Note: In my rewrite, I didn't ever say anything about $P(n)$. For example, compare the following two writeups of the same inductive proof of $n < 2^n$ for all $n \in \mathbb{Z}_{\geq 0}$.

Proof by induction (first draft – don't turn this in).

Define $P(n)$: $P(n)$ is "$n < 2^n$".
Base case: The least value of n is 0, so the base case is $P(0)$:

$$0 < 1 = 2^0. \qquad \checkmark$$

Goal: Assume $P(n)$ and show $P(n+1)$, which is

$$P(n+1): \qquad n + 1 < 2^{n+1}.$$

Inductive step: (Assume $P(n)$ and show $P(n + 1)$)
Fix $n \geq 0$ and assume $n < 2^n$ (this is the IH). Then since $n \geq 0$,

$$n + 1 \overset{\text{IH}}{<} 2^n + 1 \leq 2^n + 2^n = 2(2^n) = 2^{n+1}. \qquad \checkmark$$

Conclusion: So since $P(0)$ is true, and $P(n)$ implies $P(n + 1)$, we have $P(k)$ is true for all $k \in \mathbb{Z}_{\geq 0}$.

□

Proof (final draft). For $n = 0$, we have

$$0 < 1 = 2^0,$$

as desired. Now, fix $n \geq 0$ and assume $n < 2^n$ (for that n). Then since $n \geq 0$, we have

$$n + 1 < 2^n + 1 \leq 2^n + 2^n = 2(2^n) = 2^{n+1}.$$

Thus, the claim holds for all $n \geq 0$ by induction. □

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*where a is something like 0 or 1 or 5—whatever the lower end of the domain of the problem is.

**Don't forget to point out where you use the "Induction Hypothesis".

Problems. Prove each of the following using proof by induction. Assume throughout that n is an integer.

I. ($\star$) If $x_1, x_2, ..., x_n$ are odd integers, then their product,

$$\prod_{i=1}^{n} x_i \qquad \text{is also odd.}$$

[Note that $x_1, x_2, ..., x_n$ is an arbitrary list of odd numbers (you don't get to choose what these numbers are!).]

Hint: Since x_i is odd, for each i, we can write $x_i = 2k_i + 1$ for some $k_i \in \mathbb{Z}$. Start with $\prod_{i=1}^{1} x_i$. Then for $n \geq 1$, you have $\prod_{i=1}^{n+1} x_i = \left(\prod_{i=1}^{n} x_i\right) x_{i+1}$.

II. ($\star$) For all odd natural numbers n, $n^2 - 1$ is divisible by 8.
[Warning: Do not induct on n. Instead, start by writing what it means for n to be odd.]

Hint: Since n is odd, we can write $n = 2k + 1$ for some $k \in \mathbb{Z}$. As implied above, you don't want to induct on n; you want to induct on k, starting with $k = 0$ (careful!! if $n = 1$, then $k = 0$). Then $n^2 - 1$ is divisible by 8 is equivalent to $n^2 - 1 = 8\ell$ for some $\ell \in \mathbb{Z}$. Plug in $n = 2k + 1$ into that equation and simplify. Then move on to $n = 2(k + 1) + 1$.

III. ($\star\star$) For all $n \geq 1$ and $0 \leq x \leq \pi$, we have $\sin(nx) \leq n \sin(x)$.
[You may use the angle addition formula for $\sin(x)$.]

Hint: Expand $\sin(nx + x)$ using the angle addition formula. Then you can use the fact that $\cos(x) \leq 1$ for all x.

IV. ($\star\star$) Let $a_n = 2^{2^n} + 1$. Then for all $n \geq 2$, the last digit of a_n is 7. †
[Hint: Rephrase "last digit is 7" in terms of remainders.]

Hint: "The last digit of a_n is 7" is equivalent to $a_n = 10k + 7$ for some $k \in \mathbb{Z}_{\geq 0}$.

V. ($\star\star\star$) The *Fibonacci Numbers* $F_1, F_2, \ldots,$ are defined by the recursive rule

$$F_1 = 1 \quad F_2 = 1 \quad \text{and} \quad F_{n+1} = F_n + F_{n-1} \text{ for all } n \geq 3.$$

For all $n \geq 1$, F_{5n} is divisible by 5.
[Hint: For the induction step, let r be the remainder of F_{5n-1} when divided by 5.]

Hint: Working backwards, you have $F_{5n+5} = F_{5n+4} + F_{5n+3}$, $F_{5n+4} = F_{5n+3} + F_{5n+2}$, and so on. You will assume something about F_{5n}, so that's how far back you'll need to take that computation.

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†Fermat conjectured that for $n \geq 1$, a_n is always prime. Indeed, $a_1 = 5, a_2 = 17, a_3 = 257, a_4 = 65,537$ are all prime. However, the conjecture fails for many large values of n (Fermat had a hard time checking because a_i gets really large quickly).

VI. ($\star\star\star$) If u and v are differentiable functions of x, then

$$\frac{d^n}{dx^n} uv = \sum_{k=0}^{n} \binom{n}{k} u_k v_{n-k},$$

where $u_i = \frac{d^i}{dx^i} u$ and $v_i = \frac{d^i}{dx^i} v$.
[You will need to prove a lemma that shows $\binom{r}{s} + \binom{r}{s+1} = \binom{r+1}{s+1}$ for all $r, s \in \mathbb{Z}_{\geq 0}$, to be proven directly.]

Hint: Just try out the first few steps and see what happens. Namely,

$$\begin{aligned}
\frac{d}{dx} uv &= \left(\frac{d}{dx} u\right) v + u(\frac{d}{dx} v) = u_1 v_0 + u_0 v_1; \\
\frac{d^2}{dx^2} uv &= \frac{d}{dx} \left(u_1 v_0 + u_0 v_1\right)) \\
&= \left(\frac{d}{dx} u_1\right) v_0 + u_1(\frac{d}{dx} v_0) + \left(\frac{d}{dx} u_0\right) v_1 + u_0(\frac{d}{dx} v_1) \\
&= u_2 v_0 + u_1 v_1 + u_1 v_1 + u_0 v_2 \\
&= u_2 v_0 + 2u_1 v_1 + u_0 v_2 = \sum_{i=0}^{2} \binom{2}{i} u_i v_{2-i}; \qquad \text{and} \\
\frac{d^3}{dx^3} uv &= \frac{d}{dx} \left(\frac{d^2}{dx^2} uv\right) = \frac{d}{dx} \left(\sum_{i=0}^{2} \binom{2}{i} u_i v_{2-i}\right) \\
&= \sum_{i=0}^{2} \binom{2}{i} \frac{d}{dx} (u_i v_{2-i}) = \sum_{i=0}^{2} \binom{2}{i} (u_{i+1} v_{2-i} + u_i v_{2-i+1}) \\
&= \sum_{i=0}^{2} \binom{2}{i} u_{i+1} v_{2-i} + \sum_{i=0}^{2} \binom{2}{i} u_i v_{2-i+1} \\
&= (u_1 v_2 + 2u_2 v_1 + u_3 v_0) + (u_0 v_3 + 2u_1 v_2 + u_2 v_1) \\
&= u_0 v_3 + 3u_1 v_2 + 3u_2 v_1 + u_3 v_0.
\end{aligned}$$

Basically, this works exactly the same as the binomial theorem! To show the lemma $\binom{r}{s} + \binom{r}{s+1}$, just plug in $\binom{a}{b} = \frac{a!}{b!(a-b)!}$, and simplify.