PROOF LAB I: DIRECT PROOF

Instructions:

Step 1 (Today).
Work together with your group to finish the full set of exercises.

Step 2 (This weekend and next week).
Each person: pick 8 $\star$'s worth of exercises, and write up careful drafts of the proofs. Include key definitions and thorough reasoning—think about writing to convince a student taking Bridge next semester. Make sure a novice could follow and locate all critical components of the proof. **See portfolio guidelines for further instruction.**

Step 3 (By next Thursday).
Swap drafts with a peer editor. (Consider getting a peer's email address and sending them a pdf before class on Thursday.)

Provide written constructive criticism for your partner: Are key definitions included? Is the reasoning mathematically accurate? Are the key components of the proof easy to recognize? Is the notation clear? Would a few additional comments help a novice reader to follow more easily? Can the explanation be made more clean and concise?

Step 4 (Tuesday 10/16).
Turn in your proof second drafts, and comments from at least one peer editor on first drafts. (You will provide two versions of each proof–one first draft with comments, and one rewritten draft. Homework without peer editor comments will not be accepted.)

Note: To prove that multiple statements $X_1, X_2, \dots, X_n$ are all *equivalent*, we showed on Homework 6 #3 that you can either prove

(1) $X_1 \Leftrightarrow X_2$, $X_2 \Leftrightarrow X_3$, $\dots$, and $X_{n-1} \Leftrightarrow X_n$, or (2) $X_1 \Rightarrow X_2 \Rightarrow \cdots \Rightarrow X_n \Rightarrow X_1$.

Theorem (Binomial theorem)**.** *For all real numbers a and b, and non-negative integers n, we have*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k},$$

where $\binom{n}{k}$ = "n choose k" = $\frac{n!}{k!(n-k)!}$.

Recall that $n! = n(n-1)\cdots 2 \cdot 1$, where $0! = 1$. In particular, $\binom{n}{k}$ is the number of ways to pick a subset of size k from a set of size n.

Example: We have

$$\binom{4}{0} = \frac{4!}{0!4!} = 1, \quad \binom{4}{1} = \frac{4!}{1!3!} = 4, \quad \binom{4}{2} = \frac{4!}{2!2!} = 6, \quad \binom{4}{3} = \frac{4!}{3!1!} = 4, \quad \text{and} \quad \binom{4}{4} = \frac{4!}{4!0!} = 1.$$

So $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.

Problems. Complete each of the following using a direct proof.

I. ($\star$) Show that for positive real numbers x and y, we have $x > y$ if and only if $x^2 > y^2$.

II. ($\star$) Let x be an integer. Prove that x^n is odd if and only if x is odd.
(Hint: One direction uses the "binomial theorem", given above.)

III. ($\star\star$) Let $f : X \to Y$ be a function between two sets X and Y. Suppose A and B are subsets of X. Show that $f(A \cup B) = f(A) \cup f(B)$.

IV. ($\star\star$) Let X be a set, and let A and B be subsets of X. Prove $B - A = (A \cup B) - A$.

V. ($\star\star\star$) Suppose x and y are rational numbers such that $x < y$. Prove that there exists $z \in \mathbb{Q}$ such that $x < z < y$. That is, between every two rationals there exists another rational.

VI. ($\star\star\star$) Prove that the following statements are equivalent for A and B subsets of X.
(i) $A \subseteq B$,
(ii) $A \cap B^c = \emptyset$,
(iii) $A^c \cup B = X$.

VII. ($\star\star\star$) Prove that the following statements are equivalent.
(i) There exists unique x such that $P(x)$ is true.
(ii) There exists x such that (a) $P(x)$ holds and (b) for all y, we have $P(y) \Rightarrow y = x$.
(iii) There exists x so that for all y, $P(y)$ is true if and only if $y = x$.
(iv) There exists x such that $P(x)$ holds, and for all y and z, we have $(P(y) \wedge P(z)) \Rightarrow y = z$.

VIII. ($\star\star\star$) Let x_1, x_2, x_3, $\ldots x_n$ be an arrangement of the numbers $1, 2, 3, \ldots n$. (For example if $n = 5$ we could have $x_1 = 4, x_2 = 3, x_3 = 5, x_4 = 1, x_5 = 2$.) Prove that if n is odd, then $(x_1 - 1)(x_2 - 2)(x_3 - 3) \cdots (x_n - n)$ is even.
(Hint: Consider sums rather than products. You may use the fact that if the sum of some integers $n_1, \ldots, n_\ell$ is even, and exactly m of those are odd, then m is even.)